NEWTON, THE GEOMETER

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Abstract. We describe some of Newton's most profound geometrical discoveries, arguing that by thinking of him as a geometer we gain a deep insight into his peculiar genius. We pay particular attention to Newton's work on the *organic construction*, which deserves to be better known, being a classical geometrical construction of the Cremona transformation (1862). Newton was aware of its importance in geometry, using it to generate algebraic curves, including those with singularities.

1. Introduction

Isaac Newton was a geometer. Although he is much more widely known for the calculus, the inverse square law of gravitation, and the optics, geometry lay at the heart of his scientific thought. Geometry allowed Newton the creative freedom to make many of his astounding discoveries, as well as giving him the mathematical exactness and certainty that other methods simply could not.

In trying to understand what geometry meant to Newton we will also discuss his own geometrical discoveries, and the way in which he presented them. These were far ahead of their time. For example, it is well known that his classification of cubic curves anticipated projective geometry, and thanks to Arnol'd (1990) it is also now widely appreciated that his lemma on the areas of oval figures was an extraordinary leap two hundred years into Newton's future.

Less well known is his extraordinary work on the *organic construction*, which allowed him to perform what are now referred to as Cremona transformations to resolve singularities of plane algebraic curves.

Geometry was not a branch of mathematics, it was a way of doing mathematics, and Newton defended it fiercely, especially against Cartesian methods. We will ask why Newton was so sceptical of what most mathematicians regarded as a powerful new development. This will lead us to consider Newton's methods of curve construction, his affinity with the ancient mathematicians, and his wish to uncover the mysterious analysis supposedly underlying their work.

These were all hot topics in early modern geometry. Great controversy surrounded the questions of which problems were to be regarded as geometric and which methods might be allowable in their solution. The publication of Descartes' *Géométrie* (1637) was largely responsible for the introduction of algebraic methods and criteria, in spite of Descartes' own wishes. This threw into sharp relief the demarcation disputes which arose, originally, from the ancient focus on allowable rules of construction, and we discuss Newton's challenge to Cartesian methods[1].

It is important to note that Descartes' *Géométrie* was to some extent responsible for Newton's own early interest in mathematics, and geometry in particular[2]. It was not until the 1680s that he focussed his attention on ancient geometrical methods and became dismissive of Cartesian geometry.

This will not be a review of the excellent book (Guicciardini, 2009), but we will refer to it more than to any other. We find in this book compelling arguments for a complete re-appraisal of the core of Newton's work.

We would like to thank June Barrow-Green, Luca Chiantini, and Jeremy Gray for their help and encouragement.

[1]According to David Gregory, Newton referred to people using Cartesian methods as the "bunglers of mathematics"! See page 42 of (Hiscock, 1937).

[2]Newton studied van Schooten's second Latin edition of *Géométrie*.

2. Analysis and synthesis

As Guicciardini[3] argues, the certainty Newton sought was "guaranteed by geometry", and Newton "believed that only geometry could provide a certain and therefore publishable demonstration". But how, precisely, was geometry to be defined? In order to obtain this certainty, it was necessary to *know* and *understand* precisely what it was that was to be demonstrated. This had been a difficult question for the early modern predecessors of Newton. What did it really mean to have *knowledge* of a geometrical entity? Was it simply enough to postulate it, or to be able to deduce its existence from postulates, or should it be physically constructed, even when this is merely a representation of the object?

If it should be physically constructed, then by what means? For example, Kepler (1619) took the view[4] that only the strict Euclidean tools should be used. He therefore regarded the heptagon as "unknowable", although he was happy to discuss properties that it would have were it to exist. On the other hand, Viète (1593) believed that the ancient *neusis* construction should be adopted as an additional postulate, and showed that one could thereby solve problems involving third and fourth degree equations[5].

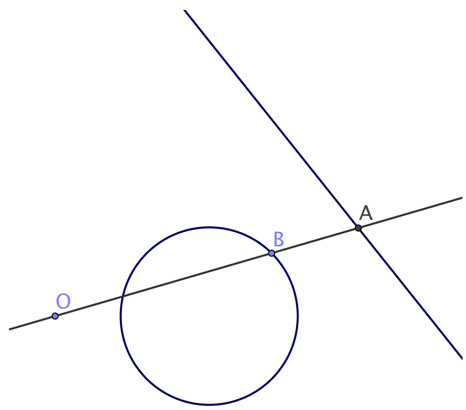

Figure 1. Neusis: given a line with the segment AB marked on it, to be able to rotate this line about O and slide it through O until A lies on the fixed line and B lies on the fixed circle.

We defer until later a discussion of Newton's preferred construction methods. [Given that *neusis* was used in ancient times, it is striking that Euclid chose arguably the most restrictive set of axioms. We are attracted by the hypothesis in (Fowler, 1999) that these were chosen because the anthyphairetic sequences which are eventually periodic are precisely those which come from ratios with ruler and compass constructions. In other words those ratios for which the Euclidean algorithm gives a finite description have ruler and compass constructions. However, this is a digression here, as there is no evidence that Newton was aware of this property of Euclidean constructions.]

The early modern mathematicians followed the ancients in dividing problem solving into two stages. The first stage, *analysis*, is the path to the discovery of a solution. Bos[6] explores in great depth the various types of analysis that may have been performed. The main distinction we shall make here is between algebraic and geometric analyses. We shall see that in the mid 1670s Newton became skeptical of the algebraic methods, and the idea of an algebraic analysis no less so.

The second stage, *synthesis*, is a demonstration of the construction or solution. This was a crucial requirement before a geometrical problem could be considered solved. Indeed, following the ancient geometers, early modern mathematicians usually removed all traces of the underlying analysis, leaving only the geometrical construction.

[3]See page 13 of (Guicciardini, 2009).
[4]See section 11.3 of (Bos, 2001).
[5]See pages 167–168 of (Bos, 2001).
[6]See chapter 5 of (Bos, 2001).

Of course, in many cases this geometrical construction was simply the reverse of the analysis, and Descartes tried to maintain this link between analysis and synthesis even when the analysis, in his case, was entirely algebraic. Newton argued, however, that this link was broken:

> Through algebra you easily arrive at equations, but always to pass therefrom to the elegant constructions and demonstrations which usually result by means of the method of porisms is not so easy, nor is one's ingenuity and power of invention so greatly exercised and refined in this analysis[7].

There are two points being made here. One is that the constructions arising from Cartesian analysis were anything but elegant, and that one should instead use the method of porisms, about which we will say more in a moment. The other is that the Cartesian procedures are algorithmic, and allow no room for the imagination.

In spite of the methods in Descartes' *Géométrie* having become widely accepted, Newton believed that there not only could but should be a geometrical analysis. Early in his studies he mastered the new algebraic methods, and only later turned his attention to classical geometry, reading the works of Euclid and Apollonius, and Commandino's Latin translation of the *Collectio* (1588) by the fourth century commentator Pappus. According to his friend Henry Pemberton (1694–1771), editor of the third edition of *Principia Mathematica*, Newton had a high regard for the classical geometers[8]:

> Of their taste, and form of demonstration Sir Isaac always professed himself a great admirer: I have heard him even censure himself for not following them yet more closely than he did; and speak with regret for his mistake at the beginning of his mathematical studies, in applying himself to the works of Des Cartes and other algebraic writers, before he had considered the elements of Euclide with that attention, which so excellent a writer deserves.

It was from Pappus' work that Newton learned of what he believed to be the ancient method of analysis: the porisms. Guicciardini explores the possibility that Newton may have been trying to somehow recreate Euclid's work on porisms in order to identify ancient geometrical analysis[9]. Agreement on precisely what the classical geometers meant by a porism is still elusive, but as the early modern geometers understood it, the porisms required the construction of a locus satisfying set conditions, such as the ancient problem that came to be known as Pappus' Problem.

3. Pappus' Problem

The contrast between Newton and Descartes is perhaps nowhere more evident than in their approaches to the *Pappus problem*. This was thought to have been introduced by Euclid, and studied by Apollonius, but it is often attributed to Pappus because the general problem, extending to any number of given lines, appeared in his *Collection* (in the fourth century). The classic case, however, is the *four-line locus*[10]:

> Given four lines and four corresponding angles, find the locus of a point such that the angled distances d_i from the point to each line maintain the constant ratio $d_1 d_2 : d_3 d_4$.

Descartes dedicated much time to the problem, reconstructing early solutions in the case with five lines[11]. In his extensive study of the problem in the *Géométrie*, Descartes introduces a coordinate system along two of the lines, and points on the locus are described by coordinates in that system. He was able to reduce the four-line problem to a single quadratic equation in two variables. Bos argues[12] that the study of Pappus' problem convinced Descartes more than anything else of the power of algebraic methods.

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[7] This dates from the 1690s. See page 102 of (Guicciardini, 2009), and page 261 of volume 7 of (Newton, 1967–1981).

[8] See page 378 of (Westfall, 1980).

[9] See page 82 of (Guicciardini, 2009).

[10] The three-line problem occurs when two of these four given lines are coincident. In the general case of many lines, the angled distances must maintain the constant ratio $d_1 \dots d_k : d_{k+1} \dots d_{2k}$ for $2k$ lines, or $d_1 \dots d_{k+1} : \alpha d_{k+2} \dots d_{2k+1}$ for $2k+1$ lines.

[11] The general solution to this is the Cartesian parabola. See sections 19.2 and 19.3 in (Bos, 2001).

[12] See chapters 19 and 23 of (Bos, 2001).

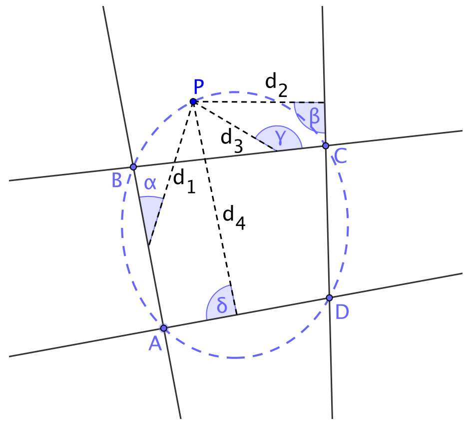

FIGURE 2. The four-line locus problem

Indeed, Descartes claimed that every algebraic curve[13] is the solution of a Pappus problem of n lines, which Newton shows to be false. Newton considered the case $n = 12$. He noted[14] that 6th degree curves have 27 parameters, whilst the corresponding Pappus problem would involve 11 or 12 lines. But the 12 line problem requires that

$$d_1 d_2 d_3 d_4 d_5 d_6 = k d_7 d_8 d_9 d_{10} d_{11} d_{12},$$

which has 22 parameters in determining the position of 11 lines with respect to the 12th, and the factor k, making 23 parameters. So, there must exist algebraic curves that are not solutions of Pappus problems.

He then develops a completely synthetic solution, in his manuscript *Solutio problematis veterum de loco solido*[15], a version of the first section of which was later included in the first edition of the *Principia*[16] (1687), Book 1 Section V, as Lemmas 17–22.

Guicciardini (2009) describes how Newton's solution is in two steps. Firstly, from Propositions 16–23 of Book 3 of the *Conics* of Apollonius[17], he shows that (in the words of Lemma 17)

> If four straight lines PQ, PR, PS, PT are drawn at given angles from any point P of a given conic to the four infinitely produced sides AB, CD, AC, DB of some quadrilateral $ABCD$ inscribed in the conic, one line being drawn to each side, the rectangle $PQ \cdot PR$ of the lines drawn to two opposite sides will be in a given ratio to the rectangle $PS \cdot PT$ of the lines drawn to the other two opposite sides.[18]

The converse is Lemma 18: if the ratio is constant then P will be on a conic. Then Lemma 19 shows how to *construct* the point P on the curve.

Newton's second step is to show how the locus which solves the problem—a conic through five given points—can be constructed. Commenting that this was essentially given by Pappus, Newton then introduces the startling *organic construction*. We will discuss this in much more detail later, but the essence is this. Newton chose two fixed points B and C called poles, and around each pole he allowed to rotate a pair of rulers, each pair at a fixed angle (the two angles not having to be equal). In each pair he designated one ruler the directing "leg" and the other the describing "leg".

There is a third special point: when the directing legs are chosen to coincide then the point of intersection of the describing legs is denoted A.

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[13]He thought of these as the geometrical curves.

[14]This dates from the late 1670s. See page 343 in volume 4 of (Newton, 1967–1981).

[15]See page 282 in volume 4 of (Newton, 1967–1981).

[16]See (Newton, 1687). Newton needed these results in this part of the *Principia* in order to find orbits of comets, but in the 1690s he considered removing them from the second edition and publishing them separately. Sections IV and V are also discussed in (Milne, 1927).

[17]Approximate dates for Apollonius are (260–190).

[18]Whiteside (1961) observes that this is equivalent to Desargues' *Conic Involution Theorem*, and also notes that the condition amounts to the constancy of a cross-ratio.

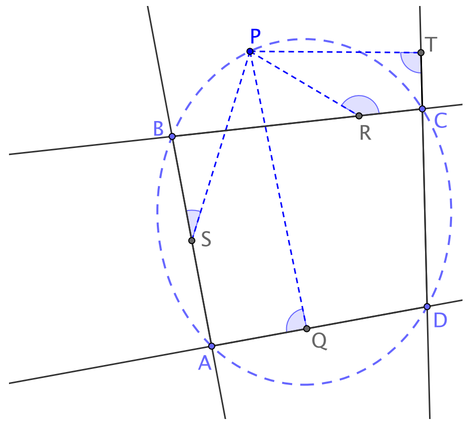

FIGURE 3. Lemma 17

In general, of course, the directing legs do not coincide, and as their point M of intersection moves, it determines the movement of the point D of intersection of the describing legs. Newton showed that if M is constrained to move along a straight line then D describes a conic through A, B, and C, and conversely that any such conic arises in this way.

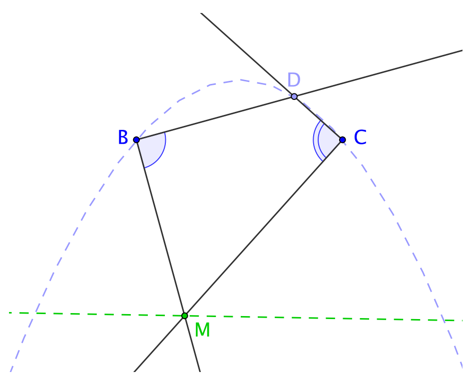

FIGURE 4. The organic construction

This beautiful result appears in the *Principia* as Lemma 21 of Book 1 Section V:

> If two movable and infinite straight lines BM and CM, drawn through given points B and C as poles, describe by their meeting-point M a third straight line MN given in position, and if two other infinite straight lines BD and CD are drawn, making given angles MBD and MCD with the first two lines at those given points B and C; then I say that the point D, where these two lines BD and CD meet, will describe a conic passing through points B and C. And conversely, if the point D, where the straight lines BD and CD meet, describes a conic passing through the given points B, C, A, and the angle DBM is always equal to the given angle ABC, and the angle DCM is always equal to the given angle ACB; then point M will lie in a straight line given in position.

Newton's proofs of both the result and its converse are elegant and clear[19]. They follow from the anharmonic property of conics (his Lemma 20) and the fact that two conics do not intersect in more than four points (his Lemma 20, Corollary 3). Guicciardini (2009) argues that this sequence of ideas came from an

[19]The point A is crucial to the construction, and it may be helpful to the reader to note that in his thesis (1961) Whiteside did not appear to grasp its importance and drew the conclusion that the proof of the converse was flawed. He corrected this misunderstanding on page 298 of volume 4 of (Newton, 1967–1981).

extension of the "main porism" of Pappus to the case of conics, and Newton had indeed been determined to restore this ancient method.

Newton's description of conics was in a fairly strong sense what we would now refer to as the projective description. In Proposition 22 he shows how to construct the conic through five given points. In fact he gives two constructions. Whiteside and others interpret the first as evidence that Newton had at least an intuitive if not explicit grasp of Steiner's Theorem[20]. The second uses the organic construction, but this should not be taken as indicating any reserve about this construction on Newton's part, as he also published it in the *Enumeratio* (1704) and the *Arithmetica Universalis* (1707).

However, in the *Principia* Newton's solution of the classical Pappus problem appears as a corollary to Lemma 19, after which he cannot resist the following comments:

> And thus there is exhibited in this corollary not an [analytical] computation but a geometrical synthesis, such as the ancients required, of the classical problem of four lines, which was begun by Euclid and carried on by Apollonius.

4. Rules for construction

> Among geometers it is in a way considered to be a considerable sin when somebody finds a plane problem by conics or line-like curves and when, to put it briefly, the solution of the problem is of an inappropriate kind[21].

The influence of this remark by Pappus was very great in the early modern period. Bos[22] gives three examples, from Descartes, Fermat, and Jacob Bernoulli, in which this passage on sin was explicitly quoted. Mathematicians wishing to extend geometrical knowledge struggled to formulate precise definitions of the subject itself, and of the simplicity of the various types ("plane", "solid", and "linear") of geometrical constructions.

It was accepted that straight lines and circles formed a basis for classical geometry, and that the way to construct them in practise was by straight edge and compasses. In addition it was also well known that the ancients had studied other curves, such as conic sections, conchoids, the Archimedian spiral, and Hippias' quadratrix, and other means of construction, such as *neusis*. However, these wider ideas were somehow less well defined than the strict Euclidean ones, and hence the focus on demarcation.

Indeed, some of these constructions were dismissed as being "mechanical", but for Descartes this did not make sense: circles and straight lines were also mechanical, in fact, and yet they were perfectly acceptable. He introduced his own "new compasses"[23] for solving the trisection problem, and wrote that the precision with which a curve could be understood should be the criterion in geometry, not the precision with which it could be traced by hand or by instruments[24].

From our point of view, Descartes' extension of the geometrical boundaries to include all algebraic curves was a dramatic and important one. Bos (2001) argues that although Descartes' attempts to define the constructions which would generate all algebraic curves were neither explicit nor conclusive, they were nevertheless the deepest part of the *Géométrie*. We describe them very briefly, and then consider Newton's fierce criticisms of them.

Descartes started by claiming that

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[20]Steiner's Theorem (1833) states that if p and p' are pencils of lines through vertices A and B respectively, and if there is a correspondence between the lines of p and p' having the property that the cross-ratio of any four lines in p is equal to the cross-ratio of the corresponding four lines in p', then the locus of the point of intersection of corresponding lines is a conic through A and B.

[21]Here, "finds" means "solves", and the strong language—sin—comes from the Latin translation of Pappus' *Collection* published by Commandino in 1588. See page 49 of (Bos, 2001).

[22]See note 31 on page 50 of (Bos, 2001).

[23]In Descartes' *Cogitationes Privatae* (1619–1620) he sketched three such instruments, one for angle trisection, and two others for solving particular cubic equations. The first was an assembly of four hinged rulers OA, OB, OC, OD, extending from a single point O. These rulers were connected by a further four rulers of fixed length, also hinged, such that the three inner angles, AOB, BOC, COD, would always be equal. These instruments certainly fulfilled Descartes' criteria for curve tracing (see below). See also section 16.4 of (Bos, 2001).

[24]See page 338 of (Bos, 2001).

> nothing else need be supposed than that two or several lines can be moved one by the other and that their intersections mark other lines

and in the interpretation by Bos these curves satisfied the four criteria

(1) the moving objects were themselves straight or curved lines,
(2) the tracing point was defined as the intersection of two such moving lines,
(3) the motions of the lines were continuous, and
(4) that they were strictly coordinated by one initial motion.

For example, Descartes objected to the quadratrix on the grounds that it required both circular and linear motions[25], which could not be strictly coordinated by one motion, because this would amount to a rectification of the circumference of a circle, which he believed "would never be known to man"[26].

This is also why Descartes rejected methods of construction in which a string is sometimes straight and sometimes curved, such as the device generating a spiral described by Huygens[27]. In contrast, he accepted pointwise constructions, but was careful to distinguish those in which generic points on the curve could be constructed, from those in which only a special subset of points on the curve could be reached. He argued that curves with these generic pointwise constructions could also be obtained by a continuous motion, so that their intersections with other similar curves could be regarded as constructible.

Having shown how to reduce the analysis of a geometrical problem to algebra, and having decided that algebraic curves were precisely those acceptable in geometry, Descartes still had to demonstrate how to perform the *synthesis*.

Descartes was faced with the task of providing the standard constructions that were to be used once the algebra had been performed. He divided problems into classes according to the degree of their equation. In each case a standard form of the equation was given, and this was to be accompanied by a standard construction. For the plane problems Descartes simply referred to the standard ruler and compass constructions, while for problems involving third and fourth degree equations he gave his own constructions using the parabola and circle. He then claimed that analogous constructions in the higher degree cases "are not difficult to find", thus dismissing the subject.

Pappus' remark depends upon having a clear criterion for the simplicity of a construction. Here Descartes adopted an unequivocally algebraic view: simplicity was defined by the degree of the equation. Guicciardini argues that Newton was in a weak position when he criticized this criterion, because Newton's arguments were based on aesthetic judgements, while Descartes' criterion was at least precise, whether right or wrong.

It is ironic that Newton's organic construction satisfied Descartes' criteria for allowable constructions, given that Newton so explicitly distanced himself from Descartes on construction methods. Newton was scornful of pointwise constructions, because one has to complete the curve by "a chance of the hand", and he also rejected, in an argument reminiscent of Kepler's[28], the "solid" constructions involving intersections of planes and cones. The underlying difference, though, was that (in modern terminology) to Descartes only algebraic curves were geometrical, the others being "mechanical", while to Newton *all* curves were mechanical:

> But these descriptions, insofar as they are achieved by manufactured instruments, are mechanical; insofar, however, as they are understood to be accomplished by the geometrical lines which the rulers in the instruments represent, they are exactly those which we embrace … as geometrical[29].

Of course, before one reaches the stage of construction, one has to perform an analysis of the problem, and here the distinction between Newton and Descartes is even clearer. For Newton, the link between analysis and construction was extremely important:

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[25]Bos shrewdly observes that "it is not necessary to pre-install a special ratio of velocities to draw a quadratrix. The ratio ... arises only because the square in which the quadratrix is to be drawn is supposed as given". See note 15 on pages 42–43 of (Bos, 2001).

[26]See page 342 of (Bos, 2001).

[27]This is from a manuscript of 1650, and Bos suggests that Huygens may have learned about this device from Descartes. See page 347 of (Bos, 2001)

[28]See page 188 of (Bos, 2001).

[29]See page 104 of (Guicciardini, 2009).

> Whence it comes that a resolution which proceeds by means of appropriate porisms is more suited to composing demonstrations than is common algebra[30].

But it was not merely a question of adopting a method which would lead to clear and elegant constructions. Newton also felt that mechanical (that is, geometrical) constructions had another crucial feature:

> [I]n definitions [of curves] it is allowable to posit the reason for a mechanical genesis, in that the species of magnitude is best understood from the reason for its genesis[31].

We note that Newton is not alone in regarding geometry as yielding deeper insights. A striking modern example comes from (Chandrasekhar, 1995). In the "Prologue" to his book Chandrasekhar says:

> The manner of my study of the *Principia* was to read the enunciations of the different propositions, construct proofs for them independently *ab initio*, and then carefully follow Newton's own demonstrations.

In his review (1995) of this book, Penrose describes Chandrasekhar's discovery that

> In almost all cases, he found to his astonishment that Newton's "archaic" methods were not only shorter and more elegant [than those using the standard procedures of modern analysis] but more revealing of the deeper issues.

5. The Organic construction

Exercitationum mathematicarum libre quinque (1656–1657), by the Dutch mathematician and commentator Frans van Schooten, includes some 'marked ruler' constructions, and a reconstruction of some of Apollonius' work *On Plane Loci*. According to Whiteside (1961), it was through a study of the fourth book, *Organica conicarum sectionum*, together with *Elementa curvarum linearum* by Schooten's student Jan de Witt[32], that Newton learnt of the organic construction.

We have seen Newton's brilliant use of the organic construction of a conic in his solution of the Pappus problem, and indeed Whiteside notes that the organic construction can, in fact, be derived almost as a corollary of Newton's work on that problem. But Newton knew that these rotating rulers could do much more: he thought of them as giving a transformation of the plane.

It was therefore natural for him to think of the construction in Lemma 21 as a transformation taking the straight line (on which the directing legs intersect) to the conic (on which the describing legs intersect). This is clear from his manuscript[33] of about 1667:

> And accordingly as the situation or nature of the line PQ varies from one place to another, so will a correspondingly varying line DE be described. Precisely, if PQ is a straight line, DE will be a conic passing through A and B; if PQ is a conic through A and B, then DE will be either a straight line or a conic (also passing through A and B). If PQ is a conic passing through A but not B and the legs of one rule lie in a straight line [. . .], DE will be a curve of the third degree [. . .][34].

In fact Newton went much further than this, as is evident for example in his lovely construction[35] of the 7-point cubic in the *Enumeratio* (1704). In this extract, note that "curves of second kind" are cubics, and that the letters do not correspond to those in our figure 5.

> All curves of second kind having a double point are determined from seven of their points given, one of which is that double point, and can be described through these same points in this way. In the curve to be described let there be given any seven points A, B, C, D, E, F, G, of which A is the double point. Join the point A and any two other of the points, say B and C, and rotate both the angle $\widehat{CAB}$ of the triangle ABC round its vertex A and either

[30]See page 102 of (Guicciardini, 2009).

[31]See page 72 of (Guicciardini, 2009).

[32]This appeared in the second edition of Schooten's translation of Descartes' *Géométrie* (1659–1661).

[33]See pages 106 and 135 of volume 2 of (Newton, 1967–1981).

[34]In this context it is interesting to note that the general problem of constructing algebraic curves by linkages was solved in (Kempe, 1876).

[35]See page 639 of volume 7 of (Newton, 1967–1981).

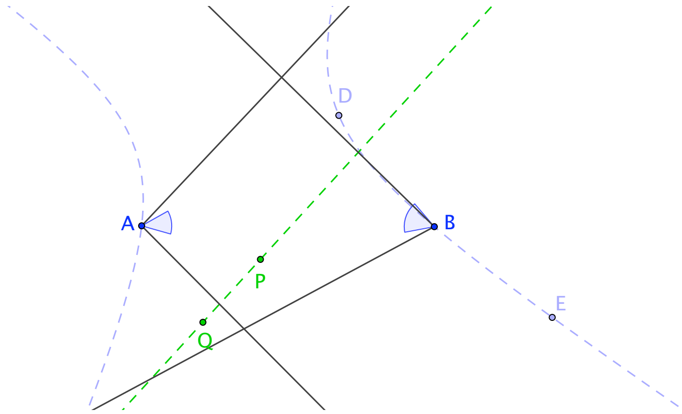

FIGURE 5. The organic construction

one, $\widehat{ABC}$, of the remaining angles round its vertex, B. And when the meeting point C of the legs AC, BC is successively applied to the four remaining points D, E, F, G, let the meet of the remaining legs AB and BA fall at the four points P, Q, R, S. Through those four points and the fifth one A describe a conic, and then so rotate the before-mentioned angles $\widehat{CAB}, \widehat{CBA}$ that the meet of the legs AB, BA traverses that conic, and the meet of the remaining legs AC, BC will by the second Theorem describe the curve proposed.

Even in his earlier manuscript (1667), Newton studied various types of singular point, and indeed he went so far as to devise a little pictorial representation of them. He also gave a long list of examples, up to and including quintics and sextics. Finally, we note that just after the construction of the 7-point cubic he considers the case in which the double point A is at infinity, as he often did elsewhere, thus in effect working in the projective plane.

As noted by Shkolenok (1972), the transformations effected by the organic construction are in fact birational maps from the projective plane to itself, now known as Cremona transformations[36]. (We give a short technical account of this in the Appendix.)

Of course one wonders how Newton could possibly have discovered such extraordinary results, so far ahead of their time, and it seems clear at least (as Guicciardini argues) that Newton actually made a set of organic rulers. For example, in the 1667 manuscript referred to above Newton uses terms such as *manufactured*, *steel nail*, and *threaded to take a nut*. Guicciardini also draws our attention to Newton's choice of language in his letter (20th of August, 1672) to Collins explaining his constructing instrument:

> And so I dispose them that they may turne freely about their poles A & B without varying the angles they are thus set at[37].

Finally, Guicciardini also notes that the drawing accompanying this letter is quite realistic. We return to this point in the next section.

6. CUBICS, AND PROJECTIVE GEOMETRY

In the early seventeenth century very little was known about cubic curves. Newton revealed the potential complexities of these curves, which, to quote Guicciardini[38] "reinforced his conviction that Descartes' criteria of simplicity were foreign to geometry". Newton's first manuscript on the subject, *Enumeratio Curvarum Trium Dimensionum*, thought to have been written around 1667, contained an equation for the general cubic

$$ay^3 + bxy^2 + cx^2y + dx^3 + ey^2 + fxy + gx^2 + hy + kx + l = 0$$

[36]These were published by Luigi Cremona in *Introduzione ad una teoria geometrica delle curve piane* Tipi Gamberini e Parmeggiani, Bologna, 1862.

[37]See page 94 of (Guicciardini, 2009)

[38]See page 112 of (Guicciardini, 2009).

which he was able to reduce to four cases by clever choices of axes.

$$\begin{aligned}
Axy^2 + By &= Cx^3 + Dx^2 + Ex + F \\
xy &= Ax^3 + Bx^2 + Cx + D \\
y^2 &= Ax^3 + Bx^2 + Cx + D \\
y &= Ax^3 + Bx^2 + Cx + D
\end{aligned}$$

He then divided the curves into 72 species by examining the roots of the right-hand side. It is often remarked that there are in fact 78 species, Newton failing to identify six of them. However, as Guicciardini points out, Newton had in fact identified the remaining six, but had chosen to omit them from his paper for some unknown reason[39]. Newton returned to his classification of cubic curves in the late 1670s with a second paper[40] *Enumeratio Linearum Tertii Ordinis* appearing as an appendix to his *Opticks* (1704).

The 1704 *Enumeratio* contained Newton's astonishing discovery that every cubic can be generated by centrally projecting one of the five divergent parabolas (encompassed by the equation $y^2 = Ax^3 + Bx^2 + Cx + D$), starting with the evocative phrase[41]

> If onto an infinite plane lit by a point-source of light there should be projected the shadows of figures ...

This remained unproved until 1731, and was first demonstrated by François Nicole (1683–1758) and Alexis Clairaut (1713–1765).

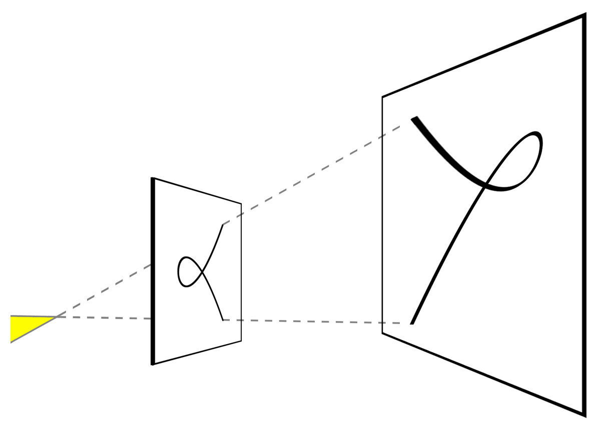

Figure 6. Projection of cubics

Here again, it seems extremely plausible that Newton's intuition was supported by his use of an actual projection from a point source of light, but Guicciardini notes that there have been differing views on this question. Rouse Ball[42] argued that the result was obtained using the projective transformations given in the *Principia*, Book 1 Section V, Lemma 22. Thus, the discovery that all the cubics can be generated by projecting the five divergent parabolas was essentially algebraic.

Talbot[43] preferred the view that Newton might have followed a geometrical procedure. He argued that Newton generated all the cubic curves by projection of the five divergent parabolas, using a method in which he began by noting that the position of the horizon line determined the nature of the asymptotes of the projected line.

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[39]See note 8 on page 111 of (Guicciardini, 2009).
[40]See volume 2 of (Whiteside, 1964–1967).
[41]See page 635 in volume 7 of (Newton, 1967–1981).
[42]See sections 6.4.2 and 6.4.3 of (Guicciardini, 2009).
[43]C R M Talbot (1803–1890) published a translation of Newton's 1704 *Enumeratio* in 1860, with notes and examples.

There is no real evidence for either hypothesis in Newton's work. Guicciardini and Whiteside both seem to favour Talbot's geometrical explanation. We agree: Newton may well have used Lemma 22 to test specific cases, but the general result must surely have been perceived by him as a geometrical insight.

7. Physics

Some of the most extraordinary examples of Newton's geometrical power arose in the exposition of his physical discoveries. In this section we note, rather briefly, three such cases, starting with a question in the foundations of the subject. Newton clearly and explicitly understood the Galilean relativity principle[44], and as was pointed out by Penrose (1987) Newton even considered[45] adopting it as one of his fundamental principles. But in what framework was this principle to operate? We agree with DiSalle, who argues[46] that Newton's absolute space and time shares with special and general relativity that space-time is an objective *geometrical* structure which expresses itself in the phenomena of motion.

Our second example comes from Section 6 of Book 1 of *Principia*, which is called *To find motions in given orbits.* Lemma 28 is on algebraically integrable ovals:

> No oval figure exists whose area, cut off by straight lines at will, can in general be found by means of equations finite in the number of their terms and dimensions.

Newton's proof simply takes a straight line rotating indefinitely about a pole inside the oval, and a point moving along the line in such a way that its distance from the pole is directly proportional to the area swept out by the line. This point describes a spiral, which intersects any fixed straight line infinitely many times.

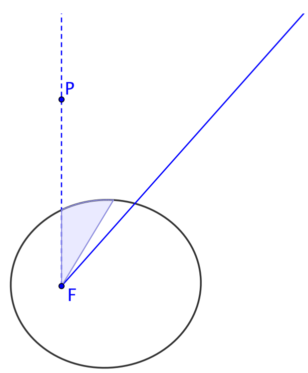

Figure 7. Lemma 28

Then, noting almost as an aside what is essentially Bézout's Theorem (1779) on the intersections of algebraic curves, the proof is completed by the observation that the if spiral were given by a polynomial then it would intersect any fixed straight line finitely many times.

At the end of his proof Newton applies the result to ellipses (which were of course the original motivation), and defines "geometrically rational" curves, noting casually that spirals, quadratrices, and cycloids are geometrically irrational. Thus he leapt to the modern demarcation of algebraic curves, while demonstrating that a restriction to these curves (following Descartes) would not be enough for a description of orbital motion.

This is how Arnol'd puts it[47]:

[44]See page 28 of (DiSalle, 2006).

[45]This is in *De motu corporum in mediis regulariter cedentibus*. See pages 188–194 in volume 6 of (Newton, 1967–1981).

[46]See page 16 of (DiSalle, 2006).

[47]See page 94 of (Arnol'd, 1990).

Comparing today the texts of Newton with the comments of his successors, it is striking how Newton's original presentation is more modern, more understandable and richer in ideas than the translation due to commentators of his geometrical ideas into the formal language of the calculus of Leibnitz.

Unfortunately, Newton did not make explicit what he meant by an oval, which has led to considerable controversy[48]. Although in later editions of the *Principia* Newton inserted a note excluding ovals "touched by conjugate figures extending out to infinity", he never made clear his assumptions on the smoothness of the oval. Nor did the statement of the Lemma distinguish between local and global integrability. There is therefore a family of possible interpretations of Newton's work, which has been elegantly dissected in (Pourciau, 2001), who concludes that:

> ... Newton's argument for the algebraic nonintegrability of ovals in Lemma 28 embodies the spirit of Poincaré: a concern for existence or nonexistence over calculation, for global properties over local, for topological and geometric insights over formulaic manipulation ...

Our final example comes from Section 12 of Book 1, which has the title *The attractive forces of spherical bodies.* Here Newton shows that the inverse square law of gravitation is not an approximation when the attracting body is a sphere instead of a point, and one of the key results is Proposition 71:

> a corpuscle placed outside the spherical surface is attracted to the centre of the sphere by a force inversely proportional to the square of its distance from that same centre.

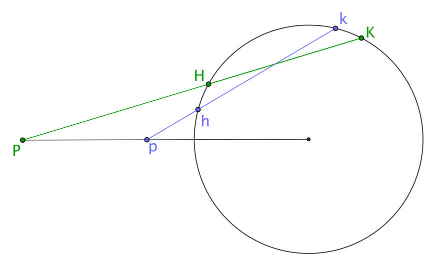

Figure 8. Gravitational attraction of a spherical shell

Newton's proof is utterly geometrical, and utterly beautiful[49]. Here is a sketch of the argument. The spherical surface attracts "corpuscles" at P and p, and we wish to find the ratio of the two attractive forces. Draw lines PHK and phk such that $HK = hk$, and draw infinitesimally close lines PIL and pil with $IL = il$. (These are not shown in our figure.) Rotate the segments HI and hi about the line Pp to obtain two ring-shaped slices of the sphere. Compare the attractions of these slices at P and p respectively, merely using the many similar triangles in the construction, and obtain the result.

Littlewood (1948) felt that the proof's key geometrical construction (of the lines PHK and phk cutting off equal chords HK and hk) "must have left its readers in helpless wonder", but conjectured that Newton had first proved the result using calculus, only later to give his geometrical proof. We agree with (Chandrasekhar, 1995) that this is highly implausible. As Chandrasekhar says, "his physical and geometrical insights were so penetrating that the proofs emerged whole in his mind"[50]. We would argue, further, that the integration Newton is supposed to have performed would in no way have suggested the key geometrical construction. In other words, there is absolutely no link between the supposed analysis and the synthesis.

[48] Whiteside's own counter-example (which he gave in note 121 on pages 302–303 in volume 6 of (Newton, 1967–1981)) was elegantly ruled out in (Pesic, 2001).

[49] It certainly meets Whitehead's criterion of style! See page 19 of A N Whitehead, *The Aims of Education and Other Essays*, New York: Macmillan, 1929

[50] Compare Penrose's discussion of this feature of inspirational thought, and his remarks on Mozart's similar ability to seize an entire composition in his mind, on page 423 of (Penrose, 1989).

8. Concluding remarks

In focussing on Newton's geometry we do not mean to imply that he was not also a brilliant algebraist, of which there is ample evidence in the *Principia*, and as we noted in our introduction he is of course widely known for his calculus.

However, it is unfortunate, to say the least, that Newton claimed that he had first found the results in the *Principia* by using the calculus, a claim for which there is no evidence at all[51].

On the contrary, many scholars have given clear and convincing arguments that Newton's claim is simply false. Guicciardini (2009) rehearses these, as do Cohen (1995) and Needham (1993), for example. The claim was made during the row with Leibnitz over priority, and simply does not make sense.

Of course the calculus was another profound achievement of Newton's, but just because the calculus came to dominate mathematics it should not be assumed that Newton must always have used it in this way. Why ever *should* he?

Newton was one of the most gifted geometers mathematics has ever seen, and this allowed him to see further, much further, than others, and to express this extraordinary insight with precision and certainty.

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[51] See page 123 of (Cohen, 1995).

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Appendix: Cremona transformations

In (Newton, 1687) Book 1 Section 5 Lemma 21 it is shown that the organic transformation maps a line to a conic through the poles B and C, and conversely that any conic through the three points B, C, and A will be mapped to a line.

The crucial part of this is that the conic goes through the point A (as well as the two poles B and C). This point A is special: it is the third of the three points which are needed for the Cremona transformations[52].

Note also that it is clear from this Lemma that the organic transformation is generically one-one and self-inverse. It can be shown by a short analytical argument that organic transformations are rational maps [53]. But a rational map is birational if and only if it is generically one-to-one[54]. So the organic transformation is a birational map from $\mathbf{P}^2$ to itself, and hence a *Cremona transformation*.

Without loss of generality we can take the points A, B, and C to have homogeneous coordinates $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Conics in $\mathbf{P}^2$ through these three points have the form

$$axy + byz + czx = 0.$$

Consider the *standard quadratic transformation* $\phi : \mathbf{P}^2 \to \mathbf{P}^2$

$$\phi(x, y, z) = (yz, zx, xy),$$

which is a special case of a Cremona transformation. Let L be a line in the codomain. Then L is

$$axy + byz + czx = 0,$$

which is a conic through $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ in the domain. So the space of lines in the codomain is the same as this linear system of conics in the domain, and $\phi^{-1}(L)$ is one of these conics.

In fact, the organic transformation is this standard quadratic transformation. To see this we use Hartshorne's argument[55], as follows.

Let $\mathcal{S}$ be the subsheaf of $\mathcal{O}(2)$ consisting of those elements which vanish at the three base points, and let

$$s_0, s_1, s_2 \in \Gamma(\mathbf{P}^2, \mathcal{S})$$

be global sections which generate $\mathcal{S}$. In other words s_0, s_1, and s_2 are three conics which generate the linear system of conics through the three base points. Also, let

$$x_0, x_1, x_2 \in \Gamma(\mathbf{P}^2, \mathcal{O}(1))$$

be global sections which generate $\mathcal{O}(1)$. Then x_0, x_1, and x_2 are simply lines generating the linear system of lines in $\mathbf{P}^2$.

Note that we are thinking of the conics as being in the *domain* $\mathbf{P}^2$, and the lines as being in the *codomain* $\mathbf{P}^2$, as in the diagram below:

[52] Newton only refers to the third base point A in the converse. In fact it is easy to see that if CA, BC, and AB intersect the line in Q, R, and S respectively, then the organic transformation maps Q to B, R to A, and S to C.

[53] We would prefer a synthetic argument for this, but have not yet found one.

[54] See page 493 of (Griffiths and Harris, 1978), for example.

[55] See page 150 of (Hartshorne, 1977).

$$\begin{array}{ccc} \mathcal{S} & & \mathcal{O}(1) \\ \downarrow & & \downarrow \\ \mathbf{P}^2 & \xrightarrow{\phi} & \mathbf{P}^2 \end{array}$$

Then there is a *unique* rational map

$$\phi : \mathbf{P}^2 \to \mathbf{P}^2$$

such that

$$\mathcal{S} = \phi^*(\mathcal{O}(1)),$$

with $s_i = \phi^*(x_i)$. In other words there is a *unique* rational map from $\mathbf{P}^2$ to itself with the property that for any line L in the codomain, $\phi^{-1}(L)$ is a conic in the domain through the three base points. So the organic transformation is the same as the standard quadratic transformation.

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