Value of Information in Social Learning*

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March 10, 2025

Abstract

This study extends Blackwell's (1953) comparison of information to a sequential social learning model, where agents make decisions sequentially based on both private signals and the observed actions of others. In this context, we introduce a new binary relation over information structures: An information structure is *more socially valuable* than another if it yields higher expected payoffs for *all* agents, regardless of their preferences. First, we establish that this binary relation is strictly stronger than the Blackwell order. Then, we provide a necessary and sufficient condition for our binary relation and propose a simpler sufficient condition that is easier to verify.

Keywords: Comparison of experiments, Social learning, Herding

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*We thank Michihiro Kandori and Ryo Shirakawa for providing insightful comments. Sato acknowledges the financial support from the JSPS KAKENHI Grant 24KJ0100. Shimizu acknowledges the financial support from the JSPS KAKENHI Grant 23KJ0667. All remaining errors are our own.

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1 Introduction

In classical decision theory, an information source is considered more valuable than another if it enables an individual decision-maker to make better choices under uncertainty. This is established by Blackwell's (1953) comparison of information structures, which evaluates information structures based on whether a single agent would always prefer one over another, regardless of their preferences.

However, in many real-world settings, decision-makers acquire information not only from their private signals but also from the observed actions of others. This creates an information externality: an individual's decision not only affects their own outcome but also serves as a source of information for future decision-makers. Because of this externality, simply comparing information structures solely based on their value for individual decision-making is no longer sufficient to evaluate the value of information structure in the society. This raises a fundamental question: *When is one information structure more socially valuable than another?*

To address this question, we extend Blackwell's (1953) comparison of information structures to the classical sequential social learning model (Banerjee, 1992; Bikhchandani et al., 1992; Smith and Sørensen, 2000). In this model, homogeneous agents make decisions sequentially based on the past actions of others and their own private signals. These private signals are drawn independently from an identical information structure depending on binary payoff-relevant states. Within this framework, we introduce a binary relation over information structures: An information structure is *more socially valuable* than another if it yields higher expected payoffs for *all* agents, regardless of their preferences, in the presence of social learning.

We first observe that our binary relation is strictly stronger than Blackwell's order (Proposition 1). This is intuitive because the history of past actions garbles signal realizations depending on the underlying decision problem. Thus, our binary relation requires a sufficiently informative signal to ensure that the joint value of history and the private signal increases. This highlights an essential feature of settings where past signals are not directly observable. If agents could observe past signal realizations instead of actions, then a Blackwell more informative signal would always be more socially valuable.

Next, Proposition 2 provides a necessary and sufficient condition for our binary relation. Specifically, one information structure is more socially valuable than another if and only if it yields higher expected payoffs for all agents, decision problems, and equilibria, even when past signals (rather than actions) are observable under the alternative information structure. The necessary condition, combined with classical results, indicates that an information structure is more socially valuable than another only if it induces unbounded (private) beliefs. Thus, if an information structure induces an information cascade, then it is no longer more socially valuable than any other information structure.

Even if one information structure induces unbounded beliefs, verifying the sufficiency part in Proposition 2 is challenging, as it depends on the underlying decision problem. To address this, we provide a clear and simple sufficient condition. Specifically, Proposition 3 states that an information structure is more socially valuable than another if it has a sufficiently high probability of revealing convincing signals.

This sufficient condition follows from the intrinsic properties of mixtures of two extreme information structures. First, our sufficient condition is equivalent to ensuring the existence of a mixture of full and no information between two information structures in Blackwell order. Moreover, under any such mixture, any equilibrium expected payoffs match those in a setting where agents observe past signals rather than actions. Finally, any such mixture respects the Blackwell order: If an information structure is Blackwell more informative than the mixture, it is also more socially valuable. If it is Blackwell dominated by the mixture, it is less socially valuable. Thus, if a mixture of full and no information exists between two information structures in Blackwell order, they are also comparable in our order.

1.1 Related Literature

Pioneered by Blackwell (1951), numerous studies have extended Blackwell's comparison of experiments.[1] Two strands of literature are closely related to our study: one focuses on game-theoretic settings (Lehrer et al., 2010, 2013; Gossner, 2000; Pęski, 2008; Cherry and Smith, 2012; Bergemann and Morris, 2016), and the other examines on large i.i.d. samples (Stein, 1951; Torgersen, 1970; Moscarini and Smith, 2002; Azrieli, 2014; Mu et al., 2021). In contrast, we extend the comparison to accommodate any number of i.i.d. draws within a game-theoretic setting without payoff externalities. The key challenge lies in garbling past signals through (coarser) actions, where the extent of this garbling crucially depends on the underlying decision problem.

Starting from Banerjee (1992); Bikhchandani et al. (1992); Smith and Sørensen (2000),[2] a fundamental question in the social learning literature is whether agents can eventually learn the true state under various settings, such as limited observations of past actions (Çelen and Kariv, 2004; Acemoglu et al., 2011; Lobel and Sadler, 2015; Arieli and Mueller-Frank, 2019, 2021; Kartik et al., 2024),[3]

[1]Some studies investigate Blackwell's comparison by restricting decision problems or experiments (Lehmann, 1988; Persico, 2000; Athey and Levin, 2018; Ben-Shahar and Sulganik, 2024), while others incorporate correlated signals or dynamic settings (Brooks et al., 2024; Renou and Venel, 2024; Whitmeyer and Williams, 2024).

[2]For a comprehensive survey, see Bikhchandani et al. (2024).

[3]See also Banerjee and Fudenberg (2004), Gale and Kariv (2003), Callander and Hörner (2009), and Smith and Sorensen (2013) for studies on observational learning where agents observe only summary statistics of past actions.

costly acquisition of private signal (Mueller-Frank and Pai, 2016; Ali, 2018), and costly observation of past actions (Kultti and Miettinen, 2006, 2007; Song, 2016; Xu, 2023).[4] To the best of our knowledge, the comparison of experiments, which is the focus of this study, remains largely unexplored in the literature. The main technical difficulty arises from the complex expression of expected payoffs without restricting attention to asymptotic agents. Our approach addresses this issue by leveraging the properties of mixtures of full and no information.

2 Model

There is an infinite sequence of ordered agents $i = 1, 2, \ldots$. Agents decide their actions sequentially. There is a binary state space $\Omega = \{L, H\}$ with a common prior. Let $\mu_0 \in (0, 1)$ be the prior of $\omega = H$. The periods are discrete ($t = 0, 1, \ldots$), and each agent i takes an action at period i from a finite action set A. A common payoff function $u : A \times \Omega \to \mathbb{R}$ determines each agent's payoff. The payoff of agent i depends solely on their own action and the state, independent of the actions taken by other agents.

The timing of this game is as follows: At period 0, nature first determines the true state, which remains unchanged in the future. In each period i, agent i first observes the actions taken by the previous agents $(1, 2, \ldots, i-1)$. Furthermore, agent i observes a private signal $s \in S$, which is drawn independently from an information structure $\pi : \Omega \to \Delta(S)$. For simplicity, we assume that S is finite.[5] Following these observations, agent i selects an action from the action set A.

Given the decision problem $\mathcal{D} = (u, A)$ and the information structure π :

[4]Other important questions include social learning with correlated information structures (Liang and Mu, 2020; Awaya and Krishna, 2025), speed/efficiency of learning (Hann-Caruthers et al., 2018; Rosenberg and Vieille, 2019), and learning about informativeness (Huang, 2024).

[5]Although the proof remains largely the same when both A and S are countable, we impose this assumption to simplify notation.

$\Omega \to \Delta(S)$, the strategy of the agent i is denoted by $\sigma_i : A^{i-1} \times S \to \Delta(A)$. Given $\mathcal{D} = (u, A)$, π, and strategy profile $\boldsymbol{\sigma} = (\sigma_i)_{i \in \mathbb{N}}$, let $\alpha^{\omega}_{\le i}(\pi, \boldsymbol{\sigma}) \in \Delta(A^i)$ be the distribution of actions taken by agents $1, 2, \ldots, i$ when the state is ω, i.e.,

$$\alpha^{\omega}_{\le i}(\boldsymbol{a}|\pi, \boldsymbol{\sigma}) = \sum_{(s_1,\ldots,s_i) \in S^i} \prod_{k=1}^{i} \pi(s_k|\omega)\sigma_k(a_k|a_1, \ldots, a_{k-1}, s_k).$$

Similarly, let $\alpha^{\omega}_i(\pi, \boldsymbol{\sigma}) \in \Delta(A)$ be the distribution of actions taken by agent i when the state is ω, i.e.,

$$\alpha^{\omega}_i(a|\pi, \boldsymbol{\sigma}) = \sum_{(a'_1,\ldots,a'_{i-1}) \in A^{i-1}} \alpha^{\omega}_{\le i}(a'_1, \ldots, a'_{i-1}, a|\pi, \boldsymbol{\sigma}).$$

Note that $\alpha^{\omega}_i(\pi, \boldsymbol{\sigma})$ does not depend on the strategies of agents after i. Let $V^{\mathcal{D}}_i(\pi, \boldsymbol{\sigma})$ be the ex-ante expected payoff for agent i. Precisely,

$$V^{\mathcal{D}}_i(\pi, \boldsymbol{\sigma}) = \mathbb{E}_{\omega}\left[\sum_{a \in A} \alpha^{\omega}_i(a|\pi, \boldsymbol{\sigma})u(a, \omega)\right].$$

We say that strategy profile $\boldsymbol{\sigma}^*$ is a Bayes-Nash Equilibrium (hereafter referred to simply as an equilibrium) under $(\mathcal{D}, \pi)$ if

$$V^{\mathcal{D}}_i(\pi, \boldsymbol{\sigma}^*) \ge V^{\mathcal{D}}_i(\pi, (\sigma_i, \sigma^*_{-i}))$$

for all σ_i and i.

For two information structures $\pi : \Omega \to \Delta(S)$ and $\pi' : \Omega \to \Delta(S')$, define their product $\pi \otimes \pi' : \Omega \to \Delta(S \times S')$ as

$$(\pi \otimes \pi')((s, s')|\omega) = \pi(s|\omega)\pi'(s'|\omega)$$

for all $s \in S$, $s' \in S'$, and $\omega \in \Omega$. We denote

$$\pi^{\otimes i} = \pi \otimes \cdots \otimes \pi.$$

as the information structure generated by i conditionally independent observations from π. Define $\overline{V_i}^{\mathcal{D}}(\pi)$ as

$$\overline{V_i}^{\mathcal{D}}(\pi) = \max_{\sigma_i : S^i \to \Delta(A)} \mathbb{E}_{\omega}\left[\sum_{a \in A} \sum_{\boldsymbol{s} \in S^i} \sigma_i(a|\boldsymbol{s})\pi^{\otimes i}(\boldsymbol{s}|\omega)u(a, \omega)\right].$$

In other words, this represents the maximized expected payoff when agent i conditionally independently observes the signal drawn from π for i times.

Given the information structure $\pi : \Omega \to \Delta(S)$, define $\mu \in \Delta[0,1]$ as the private belief distribution induced by π. More precisely, for $x \in [0,1]$,

$$\mu(x) = \sum_{s \in S(x)} [\mu_0 \pi(s|H) + (1-\mu_0)\pi(s|L)],$$

where $S(x) = \{s \in S \mid \frac{\mu_0 \pi(s|H)}{\mu_0 \pi(s|H) + (1-\mu_0)\pi(s|L)} = x\}$. For abuse of notation, define

$$\pi(\mu = x|H) = \sum_{s \in S(x)} \pi(s|H)$$

$$\pi(\mu = x|L) = \sum_{s \in S(x)} \pi(s|L).$$

We say signal s is a *conclusive signal about $\omega = H$ (resp. $\omega = L$)* if $s \in S(1)$ (resp. $s \in S(0)$). We say information structure π discloses *no information* if $\text{supp}(\mu) = \{\mu_0\}$, and π discloses *full information* if $\text{supp}(\mu) = \{0,1\}$. Given π, $\boldsymbol{\sigma}$, and $i \geq 2$, define $\rho_i \in \Delta[0,1]$ as the public belief distribution:

$$\rho_i(x) = \sum_{\boldsymbol{a} \in A^{i-1}(x)} \left(\mu_0 \alpha^H_{\leq i-1}(\boldsymbol{a}|\pi, \boldsymbol{\sigma}) + (1-\mu_0)\alpha^L_{\leq i-1}(\boldsymbol{a}|\pi, \boldsymbol{\sigma}) \right),$$

where

$$A^{i-1}(x) = \left\{ \boldsymbol{a} \in A^{i-1} \;\middle|\; \frac{\mu_0 \alpha^H_{\leq i-1}(\boldsymbol{a}|\pi, \boldsymbol{\sigma})}{\mu_0 \alpha^H_{\leq i-1}(\boldsymbol{a}|\pi, \boldsymbol{\sigma}) + (1-\mu_0)\alpha^L_{\leq i-1}(\boldsymbol{a}|\pi, \boldsymbol{\sigma})} = x \right\}.$$

Given public belief x and private belief y, the posterior belief is calculated as

$$\frac{xy}{xy + \frac{\mu_0}{1-\mu_0}(1-x)(1-y)}.$$

3 Results

We say an information structure π is *more socially valuable* than π' if for any action set A and payoff function u, the ex-ante expected payoffs for all agents in any equilibrium under π are weakly higher than those in any equilibrium under π', that is, $V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) \geq V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**})$ for any decision problem $\mathcal{D} = (u, A)$, equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$, and equilibrium $\boldsymbol{\sigma}^{**}$ under $(\mathcal{D}, \pi')$.[6] We denote $\pi \succsim_S \pi'$ when π is more socially valuable than π' and $\pi \succsim_B \pi'$ when π is Blackwell more informative than π', that is, when π' is a garbling of π.

Our first observation establishes that our binary relation is stronger than the Blackwell order, as stated below.

Proposition 1. $\succsim_S$ is a strictly stronger binary relation than $\succsim_B$.

Proof. If $\pi \succsim_S \pi'$, then $\pi \succsim_B \pi'$ as agent 1 prefers π over π' for all decision problems. We now show that there exists π, π' such that $\pi \succsim_B \pi'$ and π is not more socially valuable than π'. Let information structure $\pi : \Omega \to \Delta(\{s_0, s_1, s_2\})$ defined by $\pi(s_1|H) = 1 - \varepsilon$, $\pi(s_2|H) = \varepsilon$, $\pi(s_0|L) = 1 - \delta$, and $\pi(s_2|L) = \delta$. Suppose $\varepsilon > \delta$. Now, take $\varepsilon' \in (\varepsilon, 1)$ and define $\pi' : \Omega \to \Delta(\{s_0, s_1, s_2\})$ as $\pi'(s_1|H) = 1 - \varepsilon'$, $\pi'(s_2|H) = \varepsilon'$, $\pi'(s_0|L) = 1 - \delta$, and $\pi'(s_2|L) = \delta$. Then we have $\pi \succsim_B \pi'$. Now, consider the following decision problem $\mathcal{D} = (u, A)$: $A = \{a_0, a_1\}$, $u(a_0, H) = u(a_0, L) = 0$, $u(a_1, H) = 1 - r$, and $u(a_1, L) = -r$, where $r \in (\frac{\mu_0 \varepsilon}{\mu_0 \varepsilon + (1-\mu_0)\delta}, \min\{\frac{\mu_0 \varepsilon'}{\mu_0 \varepsilon' + (1-\mu_0)\delta}, \frac{\mu_0 \varepsilon^2}{\mu_0 \varepsilon^2 + (1-\mu_0)\delta^2}\})$.[7] Take any equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$. From Lemma 5 in Appendix, agent i's expected payoff is

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \mu_0(1 - \varepsilon^i)(1 - r).$$

Under $(\mathcal{D}, \pi')$, an equilibrium exists where agent i takes a_0 if and only if he receives s_0 or at least one agent before i takes a_0 since $\frac{\mu_0 \varepsilon'}{\mu_0 \varepsilon' + (1-\mu_0)\delta} > r$. Let $\boldsymbol{\sigma}^{**}$ denote this equilibrium strategy. In this equilibrium, the ex-ante expected payoff of agent i (≥ 2) is

$$V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**}) = \mu_0(1 - r) - (1 - \mu_0)\delta^i r.$$

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[6]We discuss a weaker version of this binary relation regarding the equilibrium selection rule in Section 4.

[7]Note that $\frac{\mu_0 \varepsilon^i}{\mu_0 \varepsilon^i + (1-\mu_0)\delta^i} > \frac{\mu_0 \varepsilon}{\mu_0 \varepsilon + (1-\mu_0)\delta}$ since $\varepsilon > \delta$.

Thus, the difference in payoffs is

$$\begin{aligned}
V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**}) - V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^{*}) =& \mu_0 \varepsilon^i (1-r) - (1-\mu_0)\delta^i r \\
=& \mu_0 \varepsilon^i \left(1 - \frac{\mu_0 \varepsilon^i + (1-\mu_0)\delta^i}{\mu_0 \varepsilon^i} r\right) \\
\geq& \mu_0 \varepsilon^i \left(1 - \frac{\mu_0 \varepsilon^2 + (1-\mu_0)\delta^2}{\mu_0 \varepsilon^2} r\right) \\
>& 0.
\end{aligned}$$

Therefore, $\pi \gtrsim_B \pi'$ but not $\pi \gtrsim_S \pi'$. □

Intuitively, this results follows because past actions provide coarser information than signal realizations. As a result, our binary relation requires the signal to be sufficiently informative to ensure that the joint value of history and the signal increases. In contrast, if agents could observe past signal realizations instead of actions, then a Blackwell more informative signal would always be more socially valuable. In the setting provided in the proof of Proposition 1, when signals are observable, the expected payoffs under π and π' are the same in this example. If past signals were observable, agent i with $s = s_2$ chooses a_1 if $s = s_2$ for all preceding agents. However, in the observable action setting, agent i with $s = s_2$ chooses a_0 if all predecessors choose a_0, even when all preceding agents receive $s = s_2$.

We characterize our binary relation as follows:

Proposition 2 (Characterization)**.** $\pi \gtrsim_S \pi'$ holds if and only if

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) \geq \overline{V}_i^{\mathcal{D}}(\pi')$$

for any decision problem $\mathcal{D}$ and any equilibrium $\boldsymbol{\sigma}^*$ under ($\mathcal{D}$,π).

By combining the classical result of Smith and Sørensen (2000), we can obtain a simple necessary condition from Proposition 2. We say that information structure π induces *unbounded beliefs* if $\mathrm{co}(\mathrm{supp}(\mu)) = [0, 1]$. Since asymptotic learning occurs under an observable signal setting, we obtain the following necessary condition:

Corollary 1 (Necessary condition)**.** Suppose that π' does not disclose no information. If $\pi \succsim_S \pi'$, then π induces unbounded beliefs.

Thus, if an information cascade occurs under some information structure, then it is no longer more socially valuable than any other information structure, except in some the trivial cases.

Even if one information structure induces unbounded beliefs, verifying the sufficiency part in Proposition 2 is challenging as it depends on each decision problem. We present a simple sufficient condition below which is easily verifiable.

Proposition 3 (Sufficient condition)**.** Suppose that π and π' satisfies

$$1 - \sum_{s \in \text{supp}(\pi')} \min\{\pi'(s|L), \pi'(s|H)\} \leq \min\{\pi(\mu = 0|L), \pi(\mu = 1|H)\}.$$

Then, $\pi \succsim_S \pi'$.

Recall that the necessary condition in Proposition 2 requires π induces unbounded beliefs if $\pi \succsim_S \pi'$. Then, a sufficient condition in Proposition 3 indicates that π is more socially valuable than π' if π assigns sufficiently high probability to private beliefs being both 0 and 1.

The formal proof is complex and is provided in the Appendix. The key step is derived by focusing on the intrinsic properties of mixtures of extreme information structures. To demonstrate this, we now present an equivalent condition for Proposition 3.

Lemma 1. There exists π'' such that $\text{supp}(\mu'') = \{0, \mu_0, 1\}$ and $\pi \succsim_B \pi'' \succsim_B \pi'$ if and only if

$$1 - \sum_{s \in \text{supp}(\pi')} \min\{\pi'(s|L), \pi'(s|H)\} \leq \min\{\pi(\mu = 0|L), \pi(\mu = 1|H)\}.$$

Thus, the sufficient condition in Proposition 3 is equivalent to ensure the existence of a mixture of full and no information such that $\pi \succsim_B \pi'' \succsim_B \pi'$.

The following Lemma 2 forms the cornerstone of the proof of Proposition 3.

Lemma 2 (Lemma 9 and Lemma 10)**.** Suppose $\pi \succsim_B \pi'' \succsim_B \pi'$ and $\text{supp}(\mu'') = \{0, \mu_0, 1\}$. Then, $\pi \succsim_S \pi'' \succsim_S \pi'$

Specifically, if an information structure is Blackwell more informative than the mixture, it is also more socially valuable (Lemma 9). Moreover, if it is Blackwell dominated by the mixture, it is less socially valuable (Lemma 10). Thus, whenever a mixture of full and no information exists between two information structures in Blackwell order, they remain comparable in our binary relation.

The proof of Lemma 2 proceeds as follows. First, as shown in Lemma 8, under any mixture of full and no information, all agents can achieve the same expected payoff as if they had observed past signal realizations. This holds for any decision problem and equilibrium, even though agents cannot directly infer their predecessors' private signals.[8] Given the above discussion, if π is Blackwell more informative than π' and π consists of a mixture of full and no information, then the expected payoff of agent i is weakly higher than that under i conditionally independent observations of π' (i.e., $\pi'^{\otimes i}$). Since past signals are always Blackwell more informative than history (Lemma 7), this expected payoff remains higher than that under π'.

Lemma 10 constructs a strategy profile under π that achieves the same equilibrium expected payoff as under π' when π' is a mixture of full and no information. Additionally, we show that this strategy profile provides a lower bound on the equilibrium payoffs under π. Intuitively, the construction follows this logic: Consider any equilibrium strategy under π. First, any other strategy weakly de-

[8]This feature is nontrivial because even a slight deviation in the support of private beliefs from that of the mixture can result in decision problems and equilibria that violate this property, as one can infer from the proof of Proposition 5 in Section 4.

creases her payoff due to the equilibrium condition. In particular, take a strategy in which agent i behaves as if she observes π' rather than π. As π' is a mixture of full and no information, such a strategy involves choosing the optimal action upon receiving a conclusive signal and mimicking agent $i-1$'s action otherwise. Given this, we further modify agent $i-1$'s strategy to follow the same one. This change decreases agent $i-1$'s expected payoff, which in turn, decreases agent i's (conditional) payoff from mimicking agent $i-1$. By repeating this process, we obtain a strategy profile that induces the lower bound of any equilibrium payoff. This lower bound coincides with the equilibrium payoff under π' when it consists of a mixture of full and no information.

4 Discussions

Our definition is too strong, particularly in relation to the equilibrium selection rule. As a result, our binary relation is not a partial order.

Proposition 4. $\pi \gtrsim_S \pi$ if and only if $\text{supp}(\mu) \subseteq \{0, \mu_0, 1\}$.

Proof. Suppose $\text{supp}(\mu) \subseteq \{0, \mu_0, 1\}$. Then, $1-\sum_{s\in\text{supp}(\pi')} \min\{\pi'(s|L), \pi'(s|H)\} = 1 - \pi(\mu = \mu_0|L) = \pi(\mu = 0|L) = \min\{\pi(\mu = 0|L), \pi(\mu = 1|H)\}$. Therefore, by Proposition 3, we have $\pi \gtrsim_S \pi$.

Now, we show that $\pi \gtrsim_s \pi$ does not hold if $\text{supp}(\mu) \not\subseteq \{0, \mu_0, 1\}$. It is suffices to show for the case where there exists some $x > \mu_0$ such that $x \in \text{supp}(\mu)$. Take $r \in [0, 1]$ that satisfies

$$x < r < \frac{x^2}{x^2 + \frac{\mu_0}{1-\mu_0}(1-x)^2}.$$

Consider the decision problem $\mathcal{D} = (u, A)$: $A = \{a_0, a_1\}$ and the payoff function is defined as $u(a_0, H) = u(a_0, L) = 0$, $u(a_1, H) = 1 - r$, and $u(a_1, L) = -r$. Take any equilibrium $\boldsymbol{\sigma}^* = (\sigma_i^*)_{i\in\mathbb{N}}$ and $s_1, s_2 \in S(x)$. Then, it follows that $\sigma_1^*(a_0|s_1) = 1$ and $\sigma_2^*(a_0|a_0, s_2) = 1$. Thus, $V_2^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*|s_1, s_2) = 0$. Additionally, we have $\overline{V}_2^{\mathcal{D}}(\pi|s_1, s_2) > 0$ since $r < \frac{x^2}{x^2 + \frac{\mu_0}{1-\mu_0}(1-x)^2}$. Note that for all $s'_1, s'_2 \in S$, $\overline{V}_2^{\mathcal{D}}(\pi|s'_1) \geq V_2^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*|s'_2)$.[9] Therefore,

$$\overline{V}_2^{\mathcal{D}}(\pi) > V_2^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*).$$

By Proposition 2, it follows that $\pi \succsim_S \pi$ does not hold. □

An alternative, less restrictive, binary relation considers a weaker notion of comparison. We say that an information structure π is *weakly more socially valuable* than π' if, for any action set A and payoff function u, there exists an equilibrium under π in which the ex-ante expected payoffs for all agents are weakly higher than those in any equilibrium under π'. We denote $\pi \succsim_W \pi'$ when π is weakly more socially valuable than π'.

Under this definition, it is straightforward to see that $\pi \succsim_W \pi$ holds. We highlight the difference between $\succsim_S$ and $\succsim_W$.

Example 1. Suppose that $\mu_0 = \frac{1}{2}$. Let $\pi : \Omega \to \Delta(\{s_0, s_1, s_2\})$ as $\pi(s_1|H) = 1 - \varepsilon$, $\pi(s_2|H) = \varepsilon$, $\pi(s_0|L) = 1 - \delta$, and $\pi(s_2|L) = \delta$. Additionally, let $\pi' : \Omega \to \Delta(\{s'_0, s'_1, s'_2\})$ as $\pi(s'_1|H) = 1 - \varepsilon'$, $\pi'(s'_2|H) = \varepsilon'$, $\pi'(s'_0|L) = 1 - \delta'$, and $\pi'(s'_2|L) = \delta'$. Assume that $\delta < \delta' < \varepsilon < \varepsilon'$. Thus, $\pi \succsim_B \pi'$ holds, as this condition is equivalent to $\varepsilon \leq \varepsilon'$ and $\delta \leq \delta'$. Moreover, we assume that $\frac{\varepsilon'}{\varepsilon'+\delta'} < \frac{\varepsilon}{\varepsilon+\delta} < \frac{\varepsilon'^2}{\varepsilon'^2+\delta'^2}$.[10]

We now construct a decision problem under which the necessary condition of Proposition 2 is violated, implying that $\pi \succsim_S \pi'$ does not hold. Consider that decision problem $\mathcal{D}$ is as follows: Let $x = \frac{\varepsilon}{\varepsilon+\delta}$. The action set is given

[9]This statement follows from the same argument as Lemma 7 in the Appendix.

[10]Note that this violates the sufficient condition of Proposition 3 as

$$1 - \sum_{s \in \text{supp}(\pi')} \min\{\pi'(s|L), \pi'(s|H)\} = 1 - \delta',$$

and

$$\min\{\pi(\mu = 0|L), \pi(\mu = 1|H)\} = 1 - \varepsilon.$$

by $A = \{a_0, a_1, a_2\}$, with payoffs specified as follows: $u(a_0, L) = u(a_0, H) = u(a_2, L) = u(a_2, H) = 0$ and $u(a_1, H) = 1 - x$, $u(a_1, L) = -x$.

Now consider equilibrium strategy $\boldsymbol{\sigma}^*$ under π such that agent 1 chooses action a_0 if $s = s_0$ or s_2 and a_1 if $s = s_1$. Given this strategy, the posterior belief of agent 2 when agent 1's action is a_0 and $s = s_2$ is $\frac{\varepsilon^2}{\varepsilon^2+\delta}$, which is lower than x. Given this, agent 2 optimally chooses action a_1 if and only if (i) $s = s_1$ or (ii) $s = s_2$ and agent 1 chooses action a_1. Thus, the expected payoff for agent 2 under this equilibrium is given by $V_2^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \frac{1}{2}(1 - \varepsilon^2)(1 - x)$.

However, under π', when agent 1 chooses action a_0 if $s' = s'_0$, a_2 if $s' = s'_2$, and a_1 if $s' = s'_1$, agent 2 can perfectly infer agent 1's private signal. By the assumption of $x = \frac{\varepsilon}{\varepsilon+\delta} < \frac{\varepsilon'^2}{\varepsilon'^2+\delta'^2}$, when agent 2 observes that agent 1 chooses action a_2 and private signal $s' = s'_2$, the optimal action is a_1. Thus, agent 2 optimally chooses action a_1 if and only if either (i) $s' = s'_1$ or (ii) $s' = s'_2$ and agent 1 chooses either action a_1 or a_2. Let $\boldsymbol{\sigma}^{**}$ denote the equilibrium strategy profile following this tie-breaking rule. Then, the expected payoff of agent 2 is

$$\begin{aligned}
V_2^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**}) = \bar{V}_2^{\mathcal{D}}(\pi')& \\
&= \frac{1}{2}(1 - \varepsilon')(1 - x) + \frac{1}{2}\varepsilon'(1 - \varepsilon')(1 - x) \\
&\qquad + \frac{1}{2}(\varepsilon'^2 + \delta'^2)\left(\frac{\varepsilon'^2}{\varepsilon'^2 + \delta'^2}(1 - x) + \frac{\delta'^2}{\varepsilon'^2 + \delta'^2}(-x)\right) \\
&= \frac{1}{2}(1 - x) - \frac{1}{2}\delta'^2 x
\end{aligned}$$

Since $V_2^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) < V_2^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**}) \iff \frac{\varepsilon}{\varepsilon+\delta} < \frac{\varepsilon'^2}{\varepsilon'^2+\delta'^2}$, it follows that $\pi \gtrsim_S \pi'$ does not hold.

Next, we establish that $\pi \gtrsim_W \pi'$. By directly constructing the equilibrium, we have a slightly more general observation:

Proposition 5. Suppose $\pi \gtrsim_B \pi'$ and $\text{supp}(\mu) = \{0, x, 1\}$ such that $|\mu_0 - x| \geq |\mu_0 - y|$ for all $y \in \text{supp}(\mu') \cap (0, 1)$, where $x, y \in [0, 1]$.[11] Then, $\pi \gtrsim_W \pi'$

[11] If $x = 0$ or $x = 1$, then $\text{supp}(\mu) = \{0, 1\}$.

By applying Proposition 5, we confirm that in this example, $\pi \succsim_W \pi'$ holds. The key feature is that under π, if agent 1 chooses action a_1 when $s = s_2$, then agent 2 can obtain the expected payoff as if she were able to observe the past signal realization. □

Beyond this example, we cannot obtain a general characterization or a simple sufficient condition for $\succsim_W$. The main difficulty arises from the tie-breaking issue across decision problems. In $\succsim_S$, the strong equilibrium selection rule allows us to sidestep these complications. Specifically, in the proof of Proposition 2, the strong requirement of the equilibrium selection rule simplifies the construction of the decision problem needed to derive the necessary condition. Moreover, the proof of Proposition 3 relies heavily on the properties of mixtures of full and no information, where these mixtures are independent of equilibrium selection rules. Thus, extending our analysis to $\succsim_W$ is not straightforward, leaving this as an avenue for future research.

Appendix

A Omitted Proofs

A.1 Preliminaries

For each $a^* \in A$ and $z \in [0, 1]$, define

$$B(a^*) = \left\{ z \in [0,1] \mid a^* \in \arg\max_{a \in A} \left[zu(a, H) + (1-z)u(a, L) \right] \right\}$$

$$B^{-1}(z) = \{a \in A \mid z \in B(a)\} = \arg\max_{a \in A} \left[zu(a, H) + (1-z)u(a, L) \right].$$

Lemma 3. Fix any $\mathcal{D} = (u, A)$. For each $a^* \in A$, $B(a^*)$ is closed interval.

Proof. Since $zu(a, H) + (1 - z)u(a, L)$ is continuous with respect to z, $B(a^*)$ is a closed set. Suppose $z_1 \in B(a^*)$ and $z_2 \in B(a^*)$. It follows that $z_1 u(a^*, H) +$

$(1-z_1)u(a^*,L) \geq z_1 u(a,H) + (1-z_1)u(a,L)$ and $z_2 u(a^*,H) + (1-z_2)u(a^*,L) \geq z_2 u(a,H) + (1-z_2)u(a,L)$ for any $a \in A$. Take any $t \in [0,1]$, then we have

$$\begin{aligned}
&[tz_1 + (1-t)z_2]u(a^*,H) + [1 - tz_1 - (1-t)z_2]u(a^*,L) \\
&= t[z_1 u(a^*,H) + (1-z_1)u(a^*,L)] + (1-t)[z_2 u(a^*,H) + (1-z_2)u(a^*,L)] \\
&\geq t[z_1 u(a,H) + (1-z_1)u(a,L)] + (1-t)[z_2 u(a,H) + (1-z_2)u(a,L)] \\
&= [tz_1 + (1-t)z_2]u(a,H) + [1 - tz_1 - (1-t)z_2]u(a,L).
\end{aligned}$$

Hence, $tz_1 + (1-t)z_2 \in B(a^*)$. □

Lemma 4. Fix any $\mathcal{D} = (u, A)$. Suppose $B^{-1}(z_1) \cap B^{-1}(z_2) \neq \emptyset$ for some $0 \leq z_1 < z_2 \leq 1$. Then, $B^{-1}(w) = B^{-1}(z_1) \cap B^{-1}(z_2)$ for all $w \in (z_1, z_2)$.

Proof. Take any $a_0 \in B^{-1}(z_1) \cap B^{-1}(z_2)$. Then $z_1 u(a_0,H) + (1-z_1)u(a_0,L) \geq z_1 u(a,H) + (1-z_1)u(a,L)$ and $z_2 u(a_0,H) + (1-z_2)u(a_0,L) \geq z_2 u(a,H) + (1-z_2)u(a,L)$ for all $a \in A$. Note that at least one inequality holds strictly if $a \notin B^{-1}(z_1) \cap B^{-1}(z_2)$. Hence, for any $w \in (z_1, z_2)$,

$$\begin{aligned}
&u(a_0,H)w + u(a_0,L)(1-w) \\
&= \frac{w - z_2}{z_1 - z_2}[z_1 u(a_0,H) + (1-z_1)u(a_0,L)] \\
&\qquad\qquad + \left(1 - \frac{w - z_2}{z_1 - z_2}\right)[z_2 u(a_0,H) + (1-z_2)u(a_0,L)] \\
&\geq \frac{w - z_2}{z_1 - z_2}[z_1 u(a,H) + (1-z_1)u(a,L)] \\
&\qquad\qquad + \left(1 - \frac{w - z_2}{z_1 - z_2}\right)[z_2 u(a,H) + (1-z_2)u(a,L)] \\
&= wu(a,H) + (1-w)u(a,L)
\end{aligned}$$

for all $a \in A$ and strict inequality holds for all $a \notin B^{-1}(z_1) \cap B^{-1}(z_2)$. Thus, $B^{-1}(w) = B^{-1}(z_1) \cap B^{-1}(z_2)$. □

Lemma 5. Suppose $(\mathcal{D}, \pi)$ satisfies $\text{supp}(\mu) \cap (x, 1) = \emptyset$ and $B^{-1}(0) = B^{-1}(x) = \{a_0\}$ for some $x \geq \mu_0$ and $a_0 \in A$. Take arbitrary equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$.

Then,

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \mu_0[(1 - p^i)u(a_1, H) + p^i u(a_0, H)] + (1 - \mu_0)u(a_0, L),$$

where $p = (1 - \pi(\mu = 1|H))$ and $a_1 \in B^{-1}(1)$.

Proof. If $a_0 \in B^{-1}(1)$, the statement holds because

$$\begin{aligned} V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) &= \mu_0 u(a_0, H) + (1 - \mu_0)u(a_0, L) \\ &= \mu_0[(1 - p^i)u(a_0, H) + p^i u(a_0, H)] + (1 - \mu_0)u(a_0, L) \\ &= \mu_0[(1 - p^i)u(a_1, H) + p^i u(a_0, H)] + (1 - \mu_0)u(a_0, L). \end{aligned}$$

Suppose $a_0 \notin B^{-1}(1)$. Take any equilibrium under π. Then, by Lemma 4, agent 1 chooses a_0 if and only if he receives $s \notin S(1)$. Agent 2 chooses an action from $B^{-1}(1)$ if she receives $s \in S(1)$ or agent 1 takes an action other than a_0 because she knows that the state is H. Notably, the public belief after observing a_0 is less than μ_0 and Lemma 4 implies that $B^{-1}(z) = \{a_0\}$ for all $z \in [0, x]$. Hence, agent 2 must choose a_0 if she receives $s \notin S(1)$ and agent 1 chooses a_0. Analogously, agent i takes action from $B^{-1}(1)$ if and only if he receives $s \in S(1)$ or at least one previous agent chooses an action other than a_0. Otherwise, agent i takes a_0. Therefore,

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \mu_0[(1 - p^i)u(a_1, H) + p^i u(a_0, H)] + (1 - \mu_0)u(a_0, L).$$

□

A.2 Proofs of Proposition 2 and Corollary 1

Lemma 6. If $\pi \gtrsim_B \pi'$ and $\rho \gtrsim_B \rho'$, then $\pi \otimes \rho \gtrsim_B \pi' \otimes \rho'$.

Proof. Suppose $\pi \gtrsim_B \pi'$ and $\rho \gtrsim_B \rho'$. Then, there exist Markov kernel γ_1 and γ_2 such that

$$\pi'(s'|\omega) = \sum_{s \in \text{supp}(\pi)} \gamma_1(s'|s)\pi(s|\omega)$$

$$\rho'(t'|\omega) = \sum_{t \in \text{supp}(\rho)} \gamma_2(t'|t)\rho(t|\omega)$$

for all $s' \in \text{supp}(\pi')$ and $t' \in \text{supp}(\rho')$. Then, we have

$$\begin{aligned}(\pi' \otimes \rho')((s', t')|\omega) =& \pi'(s'|\omega)\rho'(t'|\omega) \\ =& \sum_{s \in \text{supp}(\pi)} \gamma_1(s'|s)\pi(s|\omega) \sum_{t \in \text{supp}(\rho)} \gamma_2(t'|t)\rho(t|\omega) \\ =& \sum_{(s,t) \in \text{supp}(\pi \otimes \rho)} \gamma_1(s'|s)\gamma_2(t'|t)\pi(s|\omega)\rho(t|\omega) \\ =& \sum_{(s,t) \in \text{supp}(\pi \otimes \rho)} \gamma((s', t')|(s, t))(\pi \otimes \rho)((s, t)|\omega),\end{aligned}$$

where $\gamma((s', t')|(s, t)) = \gamma_1(s'|s)\gamma_2(t'|t)$. Since γ is a Markov kernel, $\pi' \otimes \rho'$ is a garbling of $\pi \otimes \rho$. □

Lemma 7.

$$\overline{V}_i^{\mathcal{D}}(\pi) \geq V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma})$$

for all $i, \mathcal{D}, \pi,$ and $\boldsymbol{\sigma}$.

Proof of Lemma 7. Take any $\mathcal{D}, \pi,$ and $\boldsymbol{\sigma}$. Note that $\overline{V}_1^{\mathcal{D}}(\pi) = V_1^{\mathcal{D}}(\pi, \boldsymbol{\sigma})$. Fix $i \geq 2$. For each $s \in S^{i-1}$, define $f_{i-1}(s) \in \Delta(A^{i-1})$ as

$$f_{i-1}(\boldsymbol{a}|s) = \prod_{k=1}^{i-1} \sigma_k(a_k|a_1, \ldots, a_{k-1}, s_k).$$

Hence, $f_{i-1}(a|s)$ is the probability that agent 1 to agent $i-1$ takes action $\boldsymbol{a} = (a_1, \ldots, a_{i-1})$ when agent 1 to agent $i-1$ receives private signal $\boldsymbol{s} = (s_1, \ldots, s_{i-1})$. Then,

$$\begin{aligned}\alpha^{\omega}_{\leq i-1}(\boldsymbol{a}|\pi, \boldsymbol{\sigma}) &= \sum_{\boldsymbol{s} \in S^{i-1}} \prod_{k=1}^{i-1} \sigma_k(a_k|a_1, \ldots, a_{k-1}, s_k)\pi(s_k|\omega) \\ &= \sum_{\boldsymbol{s} \in S^{i-1}} f_i(\boldsymbol{a}|\boldsymbol{s})\pi^{\otimes i-1}(\boldsymbol{s}|\omega).\end{aligned}$$

Thus, $\alpha_{\leq i-1}(\cdot|\pi, \boldsymbol{\sigma})$ is a garbling of $\pi^{\otimes i-1}$. By Lemma 6, we have

$$\pi^{\otimes i-1} \otimes \pi \succsim_B \alpha_{\leq i-1}(\cdot|\pi, \boldsymbol{\sigma}) \otimes \pi.$$

Hence,

$$\overline{V}_i^{\mathcal{D}}(\pi) \geq V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}).$$

□

Proof of Proposition 2. Since

$$\overline{V}_i^{\mathcal{D}}(\pi') \geq V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}')$$

for all strategy profile $\boldsymbol{\sigma}'$ by Lemma 7, $\pi \gtrsim_S \pi'$ holds if $V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) \geq \overline{V}_i^{\mathcal{D}}(\pi')$.

Conversely, suppose $\pi \gtrsim_S \pi'$. Take any $\mathcal{D} = (u, A)$, equilibrium $\boldsymbol{\sigma}^*$ under $\pi : \Omega \to \Delta(S)$, and equilibrium $\boldsymbol{\sigma}^{**}$ under $\pi' : \Omega \to \Delta(S')$. Then, $V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) \geq V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**})$ by $\pi \gtrsim_S \pi'$. Consider the decision problem $\bar{\mathcal{D}} = (\bar{A}, \bar{u})$, where $\bar{A} = \{(a, k) \mid a \in A, k \in S'\}$ and $\bar{u}((a, k), \omega) = u(a, \omega)$ for all $a \in A, \omega \in \Omega$. Fix $s_1 \in S'$ and define strategy profile $\boldsymbol{\sigma} = (\sigma_i)_{i \in \mathbb{N}}$ under $(\bar{\mathcal{D}}, \pi)$ as following:

$$\begin{cases} \sigma_i((a, s_1)|(a_1, k_1), (a_2, k_2), \ldots, (a_{i-1}, k_{i-1}), s) = \sigma_i^*(a|a_1, a_2, \ldots, a_{i-1}, s) \\ \sigma_i((a, k)|(a_1, k_1), (a_2, k_2), \ldots, (a_{i-1}, k_{i-1}), s) = 0, \end{cases}$$

for all $a \in A$, $s \in S$, $(a_1, \ldots, a_{i-1}) \in A^{i-1}$, $k_1, k_2, \ldots, k_{i-1} \in S'$, and $k \in S' \setminus \{s_1\}$. Note that $\boldsymbol{\sigma}$ is an equilibrium under $(\bar{\mathcal{D}}, \pi)$. Moreover, it follows that

$$V_i^{\bar{\mathcal{D}}}(\pi, \boldsymbol{\sigma}) = V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*).$$

Under $(\bar{\mathcal{D}}, \pi')$, if we consider the following equilibrium $\boldsymbol{\sigma}'$, the expected playoff of agent i at equilibrium ($V_i^{\bar{\mathcal{D}}}(\pi', \boldsymbol{\sigma}')$) coincides with $\overline{V}_i^{\mathcal{D}}(\pi')$. Specifically, each agent i chooses an action that maximizes his expected payoff on the equilibrium path, but always chooses an action of the form (a, k) ($a \in A$) when the received signal is $k \in S'$. Since each agent can observe signals received by their predecessor on the equilibrium path, it follows that $V_i^{\bar{\mathcal{D}}}(\pi', \boldsymbol{\sigma}') = \overline{V}_i^{\bar{\mathcal{D}}}(\pi') = \overline{V}_i^{\mathcal{D}}(\pi')$. Therefore,

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = V_i^{\bar{\mathcal{D}}}(\pi, \boldsymbol{\sigma}) \geq V_i^{\bar{\mathcal{D}}}(\pi', \boldsymbol{\sigma}') = \overline{V}_i^{\mathcal{D}}(\pi').$$

□

Proof of Corollary 1. Prove by contradiction. Suppose $\text{co}(\text{supp}(\mu)) \neq [0,1]$. Then, either $1 \notin \text{supp}(\mu)$ or $0 \notin \text{supp}(\mu)$. By symmetry, it suffices to consider a case $1 \notin \text{supp}(\mu)$. Since $\text{supp}(\mu)$ is a closed set, there exists $r \in [\mu_0, 1)$ such that $\text{supp}(\mu) \subseteq [0, r]$. Consider the following decision problem $\mathcal{D} = (u, A)$: $A = \{a_1, a_2\}$, $u(a_1, L) = u(a_1, H) = 0$, $u(a_2, H) = 1 - r$ and $u(a_2, L) = -r$. Then, the strategy profile $\boldsymbol{\sigma}^*$ that all agents always choose a_1 is an equilibrium under $(\mathcal{D}, \pi)$. It follows that

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = 0.$$

Since π' does not disclose no information, repeated observations of π' allow agents to learn the state in the limit. Hence,

$$\overline{V}_i^{\mathcal{D}}(\pi') > 0$$

for large enough i. By Proposition 2, π is not more socially valuable than π'. □

A.3 Proof of Proposition 3

The following lemma shows that the expected payoff under the mixture of full and no information is the same to the one under observable signal setting for any decision problem and equilibrium.

Lemma 8. Suppose $\text{supp}(\mu) = \{0, \mu_0, 1\}$. Fix the decision problem $\mathcal{D} = (u, A)$. Take arbitrary equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$. Then,

$$\begin{aligned} V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) &= \overline{V}_i^{\mathcal{D}}(\pi) \\ &= \mu_0[(1 - p^i)U_1] + (1 - \mu_0)[(1 - p^i)U_0] + p^i U_{\mu_0}, \end{aligned}$$

where $U_1 = \max_a u(a, H)$, $U_0 = \max_a u(a, L)$, $U_{\mu_0} = \max_a[\mu_0 u(a, H) + (1 - \mu_0)u(a, L)]$, and $p = \pi(\mu = \mu_0|H) = \pi(\mu = \mu_0|L)$.

Proof. First, it is easily calculated that

$$\overline{V}_i^{\mathcal{D}}(\pi) = \mu_0[(1-p^i)U_1] + (1-\mu_0)[(1-p^i)U_0] + p^i U_{\mu_0}.$$

To show that $V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \overline{V}_i^{\mathcal{D}}(\pi)$, we divide the decision problem into five cases.

Case (i): $B^{-1}(0) \cap B^{-1}(\mu_0) = \emptyset$ and $B^{-1}(\mu_0) \cap B^{-1}(1) = \emptyset$. Take any equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$. Then, in $\boldsymbol{\sigma}^*$, agent i chooses an action from $B^{-1}(0)$ if and only if he receives a conclusive signal about $\omega = L$ or at least one agent before i takes an action from $B^{-1}(0)$. Similarly, agent i chooses an action from $B^{-1}(1)$ if and only if he receives a conclusive signal about $\omega = H$ or at least one agent before i takes an action from $B^{-1}(1)$. Otherwise, he chooses an action from $B^{-1}(\mu_0)$ because the posterior is always μ_0. Hence, we have

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \mu_0[(1-p^i)U_1] + (1-\mu_0)[(1-p^i)U_0] + p^i U_{\mu_0}.$$

Case (ii): $B^{-1}(0) \cap B^{-1}(\mu_0) \cap B^{-1}(1) \neq \emptyset$. In this case, it is always optimal for agent i to take an action $a^* \in B^{-1}(0) \cap B^{-1}(\mu_0) \cap B^{-1}(1)$. Hence,

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \mu_0 U_1 + (1-\mu_0)U_0$$

for all equilibrium $\boldsymbol{\sigma}^*$. Since $U_{\mu_0} = \mu_0 U_1 + (1-\mu_0)U_0$, we have

$$\begin{aligned} V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) &= \mu_0[(1-p^i)U_1] + (1-\mu_0)[(1-p^i)U_0] + p^i(\mu_0 U_1 + (1-\mu_0)U_{\mu_0}) \\ &= \mu_0[(1-p^i)U_1] + (1-\mu_0)[(1-p^i)U_0] + p^i U_{\mu_0}. \end{aligned}$$

Case (iii): $B^{-1}(0) \cap B^{-1}(\mu_0) = \emptyset$ and $B^{-1}(\mu_0) \cap B^{-1}(1) \neq \emptyset$. Show that, in any equilibrium $\boldsymbol{\sigma}^*$, agent i's posterior is always in $\{0\} \cup [\mu_0, 1]$ and agent i chooses an action from $B^{-1}(0)$ if and only if agent i or at least one of the previous ones receives a conclusive signal about $\omega = L$. Prove by induction. Agent 1's posterior is in $\{0, \mu_0, 1\}$ and he takes an action from $B^{-1}(0)$ if and only if he receives a conclusive signal about $\omega = L$. Suppose the statement holds for $i = 1, 2, \ldots, k$. Consider agent $k+1$. If she receives a conclusive signal about $\omega = L$ or at least one agent before $k+1$ takes an action from $B^{-1}(0)$, she knows $\omega = L$ and takes an action from $B^{-1}(0)$. Otherwise, the public belief of agent $k+1$ is more than or equal to μ_0 from the assumption, and the posterior of agent $k+1$ is also more than or equal to μ_0. By Lemma 3, we know that $B^{-1}(0) \cap B^{-1}(z) = \emptyset$ for all $z \in [\mu_0, 1]$. Hence, the statement holds for agent $k+1$. Therefore, one of the optimal strategies for agent i is to take an action $a_0 \in B^{-1}(0)$ if she receives a conclusive signal about $\omega = L$ or at least one agent before $k+1$ takes an action from $B^{-1}(0)$, and to take an action $a_1 \in B^{-1}(\mu_0) \cap B^{-1}(1)$ otherwise. Hence, it follows that

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \mu_0 U_1 + (1-\mu_0)[(1-p^i)U_0 + p^i u(a_1, L)].$$

Since $U_{\mu_0} = \mu_0 U_1 + (1-\mu_0)u(a_1, L)$, we have

$$\begin{aligned} V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) &= \mu_0 U_1 + (1-\mu_0)(1-p^i)U_0 + p^i(U_{\mu_0} - \mu_0 U_1) \\ &= \mu_0[(1-p^i)U_1] + (1-\mu_0)[(1-p^i)U_0] + p^i U_{\mu_0}. \end{aligned}$$

Case (iv): $B^{-1}(0) \cap B^{-1}(\mu_0) \neq \emptyset$ and $B^{-1}(\mu_0) \cap B^{-1}(1) = \emptyset$. This case is omitted due to symmetry with Case (iii).

Case (v): $B^{-1}(0) \cap B^{-1}(\mu_0) \neq \emptyset$, $B^{-1}(\mu_0) \cap B^{-1}(1) \neq \emptyset$, and $B^{-1}(0) \cap B^{-1}(\mu_0) \cap B^{-1}(1) = \emptyset$. Fix arbitrary equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$. We further divide it into two cases.

Case (v-i): Suppose that, in $\boldsymbol{\sigma}^*$, no agent preceding agent i takes an action from $B^{-1}(0) \cup B^{-1}(1)$. Then, it means that all agent before i receives a signal that induces private belief μ_0. Therefore, it is only optimal for agent i to take an action from $B^{-1}(0)$ if he receives a conclusive signal about $\omega = L$, from $B^{-1}(1)$ if he receives a conclusive signal about $\omega = H$, and from $B^{-1}(\mu_0)$ otherwise.

Case (v-ii): Suppose that at least one of the previous agents takes an action from $B^{-1}(0) \cup B^{-1}(1)$. Let k be the number of the first agent who took action from $B^{-1}(0) \cup B^{-1}(1)$ and a_k be the action taken by agent k. Then, by the similar argument as before, it can be shown that agent i's posterior is always in

$\{0\} \cup (\mu_0, 1]$ if $a_k \in B^{-1}(1)$ and is always in $[0, \mu_0) \cup \{1\}$ if $a_k \in B^{-1}(0)$. Hence, it is the only optimal strategy for agent i to take an action from $B^{-1}(0)$ if agent i or at least one agent before i takes an action from $B^{-1}(0)$ and to take an action from $B^{-1}(1)$ otherwise when $a_k \in B^{-1}(1)$. Moreover, it is the only optimal strategy for agent i to take an action from $B^{-1}(1)$ if agent i or at least one agent before i takes an action from $B^{-1}(1)$ and to take an action from $B^{-1}(0)$ otherwise when $a_k \in B^{-1}(0)$.[12] We further divide into two cases.

Case (v-ii-i): From the above discussion, when the true state is $\omega = H$, if at least one agent receives a conclusive signal about $\omega = H$, agent i necessarily chooses an action from $B^{-1}(1)$ in equilibrium. This is because if $a_k \in B^{-1}(1)$, then all agents after agent k must choose an action from $B^{-1}(1)$, and if $a_k \in B^{-1}(0)$, then if one agent receives a conclusive signal about $\omega = H$ and takes action from $B^{-1}(1)$, the posterior of the subsequent agents will be 1. Similarly, when the true state is $\omega = L$, if at least one agent receives a conclusive signal, agent i necessarily chooses an action from $B^{-1}(0)$ in equilibrium.

Case (v-ii-ii): Finally, if no agent receives a conclusive signal, agent i necessarily chooses an action from $B^{-1}(\mu_0)$ in equilibrium. This is because if $a_k \in B^{-1}(0)$ then the posterior of agent i is in $(0, \mu_0)$, the optimal action is $B^{-1}(0) \cap B^{-1}(\mu_0) \subseteq B^{-1}(\mu_0)$ and if $a_k \in B^{-1}(1)$, the posterior of agent i is in $(\mu_0, 1)$, so the optimal action is $B^{-1}(1) \cap B^{-1}(\mu_0) \subseteq B^{-1}(\mu_0)$.

Thus, by Case (v-i) and Case (v-ii-ii), if no agent in $1, \ldots, i$ receive the conclusive signals, agent i chooses an action from $B^{-1}(\mu_0)$. Additionally, by Case (v-ii-i), if the true state is $\omega = H$ (resp. $\omega = L$) and at least one agent in $1, \ldots, i$ receives a conclusive signal about $\omega = H$ (resp. $\omega = L$), agent i chooses an action from $B^{-1}(1)$ (resp. $B^{-1}(0)$). Therefore, we have

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \mu_0[(1 - p^i)U_1] + (1 - \mu_0)[(1 - p^i)U_0] + p^i U_{\mu_0}.$$

[12] Here we use Lemma 4. Note that if the posterior is in $(\mu_0, 1)$, the optimal action is $B^{-1}(\mu_0) \cap B^{-1}(1) \subseteq B^{-1}(1)$.

□

Utilizing Lemma 8 and Blackwell's theorem, we can show that the expected payoff under π is weakly higher than the upper bound under π' for any decision problems if π consists of a mixture of full and no information.

Lemma 9. Suppose $\pi \gtrsim_B \pi'$ and $\text{supp}(\mu) = \{0, \mu_0, 1\}$. Then, $\pi \gtrsim_S \pi'$.

Proof. Take any $\mathcal{D} = (u, A)$. Take arbitrary equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$. From Lemma 8, we have

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \overline{V}_i^{\mathcal{D}}(\pi).$$

Hence, an expected payoff of agent i in any equilibrium is the same as an expected payoff of agent i when agent i can observe not actions taken by past agents but signals received by past agents.

Next, take any equilibrium $\boldsymbol{\sigma}^{**}$ under $(\mathcal{D}, \pi')$. Note that

$$\overline{V}_i^{\mathcal{D}}(\pi') \geq V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**})$$

by Lemma 7. Since $\pi^{\otimes i} \gtrsim_B \pi'^{\otimes i}$ by Lemma 6,

$$\overline{V}_i^{\mathcal{D}}(\pi) \geq \overline{V}_i^{\mathcal{D}}(\pi').$$

Hence, it follows that

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \overline{V}_i^{\mathcal{D}}(\pi) \geq \overline{V}_i^{\mathcal{D}}(\pi') \geq V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**}).$$

Thus, $\pi \gtrsim_S \pi'$. □

We then construct a strategy profile under π such that it achieves the same equilibrium expected payoff under π' when π' consists of a mixture of full and no information. Additionally, we show that it provides a lower bound of payoffs for all agents under π.

Lemma 10. Suppose that π and π' satisfies $\text{supp}(\mu') = \{0, \mu_0, 1\}$ and $\min\{\pi(\mu = 0|L), \pi(\mu = 1|H)\} \geq p$, where $p = \pi'(\mu' = 0|L) = \pi'(\mu' = 1|H)$. Then, $\pi \gtrsim_S \pi'$.

Proof. Let $q_L = \frac{\pi'(\mu'=0|L)}{\pi(\mu=0|L)}$ and $q_H = \frac{\pi'(\mu'=1|H)}{\pi(\mu=1|H)}$. Take any $\mathcal{D}$ and define $\boldsymbol{\sigma}^{**} = (\sigma_i^{**})_{i\in\mathbb{N}}$ as the following strategy under $(\mathcal{D}, \pi)$. Agent 1 chooses $a_0 \in B^{-1}(0)$ with probability q_L and chooses $a_2 \in B^{-1}(\mu_0)$ with probability $1 - q_L$ if he receives conclusive signal about $\omega = L$. Agent 1 chooses $a_1 \in B^{-1}(1)$ with probability q_H and chooses a_2 with probability $1 - q_H$ if he receives conclusive signal about $\omega = H$. Otherwise, agent 1 chooses a_2. For $i \geq 2$, agent i chooses a_0 with probability q_L and chooses the same action as agent $i - 1$ with probability $1 - q_L$ if he receives a conclusive signal about $\omega = L$. Agent i chooses a_1 with probability q_H and chooses the same action as agent $i - 1$ with probability $1 - q_H$ if he receives a conclusive signal about $\omega = H$. Otherwise, agent i chooses the same action as agent $i - 1$. First, note that

$$\begin{aligned} V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^{**}) &= \mu_0[(1 - p^i)U_1] + (1 - \mu_0)[(1 - p^i)U_0] + p^i U_{\mu_0} \\ &= \overline{V}_i^{\mathcal{D}}(\pi'), \end{aligned}$$

where the last equality comes from Lemma 8.

Fix an equilibrium σ^* under $(\mathcal{D}, \pi)$ and define $\boldsymbol{\sigma}(k)$ as

$$\boldsymbol{\sigma}(k) = (\sigma_1^*, \sigma_2^*, \ldots, \sigma_k^*, \sigma_{k+1}^{**}, \sigma_{k+2}^{**}, \ldots).$$

Show that if $i \geq k + 1$,

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}(k)) = \mu_0 p U_1 + (1 - \mu_0) p U_0 + (1 - p) V_{i-1}^{\mathcal{D}}(\pi, \boldsymbol{\sigma}(k)).$$

Note that

$$\begin{aligned} V_{i-1}^{\mathcal{D}}(\pi, \boldsymbol{\sigma}(k)) =& \mu_0 \sum_{a\in A} \alpha_{i-1}^H(a|\pi, \boldsymbol{\sigma}(k)) u(a, H) \\ & + (1 - \mu_0) \sum_{a\in A} \alpha_{i-1}^L(a|\pi, \boldsymbol{\sigma}(k)) u(a, L). \end{aligned}$$

Since $i \geq k + 1, \boldsymbol{\sigma}(k)_i = \sigma_i^{**}$. Hence,

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}(k)) = \mu_0[\pi(\mu = 1|H) q_H U_1 + (1 - \pi(\mu = 1|H) q_H) \sum_{a\in A} \alpha_{i-1}^H(a|\pi, \boldsymbol{\sigma}(k)) u(a, H)]$$

$$+ (1-\mu_0)[\pi(\mu=0|L)q_L U_0 + (1-\pi(\mu=0|L)q_L)\sum_{a\in A}\alpha^L_{i-1}(a|\pi,\boldsymbol{\sigma}(k))u(a,L)]$$

$$= \mu_0[pU_1 + (1-p)\sum_{a\in A}\alpha^H_{i-1}(a|\pi,\boldsymbol{\sigma}(k))u(a,H)]$$

$$+ (1-\mu_0)[pU_0 + (1-p)\sum_{a\in A}\alpha^L_{i-1}(a|\pi,\boldsymbol{\sigma}(k))u(a,L)]$$

$$= \mu_0 pU_1 + (1-\mu_0)pU_0 + (1-p)V^{\mathcal{D}}_{i-1}(\pi,\boldsymbol{\sigma}(k)).$$

By the definition of $\boldsymbol{\sigma}(k)$,

$$\begin{cases} V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}^*) = V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k)) \text{ if } i < k+1 \\ V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}^*) \geq V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k)) \text{ if } i = k+1 \end{cases}$$

The second inequality is held by the optimality of σ^*_i. We now show that

$$V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k+1)) \geq V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k))$$

for all i,k. First, if $k \geq i-1$, we have $V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k+1)) = V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}^*) \geq V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k))$. Next, we have $V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-1)) \geq V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-2))$ for $i \geq 2$ since

$$\begin{aligned} V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-2)) &= \mu_0 pU_1 + (1-\mu_0)pU_0 + (1-p)V^{\mathcal{D}}_{i-1}(\pi,\boldsymbol{\sigma}(i-2)) \\ &\leq \mu_0 pU_1 + (1-\mu_0)pU_0 + (1-p)V^{\mathcal{D}}_{i-1}(\pi,\boldsymbol{\sigma}(i-1)) \\ &= V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-1)). \end{aligned}$$

Then, we have $V_i(\pi,\boldsymbol{\sigma}(i-2)) \geq V_i(\pi,\boldsymbol{\sigma}(i-3))$ for $i \geq 3$ since

$$\begin{aligned} V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-3)) &= \mu_0 pU_1 + (1-\mu_0)pU_0 + (1-p)V^{\mathcal{D}}_{i-1}(\pi,\boldsymbol{\sigma}(i-3)) \\ &\leq \mu_0 pU_1 + (1-\mu_0)pU_0 + (1-p)V^{\mathcal{D}}_{i-1}(\pi,\boldsymbol{\sigma}(i-2)) \\ &= V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-2)). \end{aligned}$$

Analogously, it follows that $V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-m)) \geq V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(i-m-1))$ for all i,m that satisfies $i-m-1 \geq 0$. Hence,

$$V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k+1)) \geq V^{\mathcal{D}}_i(\pi,\boldsymbol{\sigma}(k))$$

for all i, k. Therefore, we have

$$\begin{aligned} V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) &= V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}(i)) \\ &\geq V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}(0)) \\ &= V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^{**}) \\ &= \overline{V_i}^{\mathcal{D}}(\pi') \end{aligned}$$

□

Proof of Lemma 1. Suppose $\mathrm{supp}(\mu'') = \{0, \mu_0, 1\}$. Then $\pi \gtrsim_B \pi''$ is equivalent to

$$\begin{aligned} &\pi(\mu = 0|L) \geq \pi''(\mu = 0|L) \textit{ and} \\ &\pi(\mu = 1|H) \geq \pi''(\mu = 1|H). \end{aligned}$$

Show that $\pi'' \gtrsim_B \pi'$ is equivalent to

$$\begin{aligned} \pi''(\mu = \mu_0|L) = \pi''(\mu = \mu_0|H) \\ \leq \sum_{s \in S'} \min\{\pi'(s|L), \pi'(s|H)\}. \end{aligned}$$

Suppose $\pi''(\mu = \mu_0|L) = \pi''(\mu = \mu_0|H) \leq \sum_{s \in S'} \min\{\pi'(s|L), \pi'(s|H)\}$. Define $\rho : \Omega \longrightarrow \Delta\{s_0, s_1, s_2\}$ that satisfies

$$\begin{aligned} &\rho(s_1|L) = 0 \\ &\rho(s_0|H) = 0 \\ &\rho(s_2|H) = \rho(s_2|L) = \sum_{s \in S'} \min\{\pi'(s|L), \pi'(s|H)\}. \end{aligned}$$

Then, we have $\pi'' \gtrsim_B \rho$ as $\mathrm{supp}(\mu'') = \{0, \mu_0, 1\}$.

If $\rho(s_2|L) = \rho(s_2|H) = 1$, $\rho \gtrsim_B \pi'$ as π' discloses no information. If $\rho(s_2|L) = \rho(s_2|H) = 0$, $\rho \gtrsim_B \pi'$ as both ρ and π' disclose full information. Otherwise,

$$\begin{aligned} \pi'(s|\omega) = &\frac{\max\{\pi'(s|L) - \pi'(s|H), 0\}}{\rho(s_0|L)}\rho(s_0|\omega) + \frac{\max\{\pi'(s|H) - \pi'(s|L), 0\}}{\rho(s_1|H)}\rho(s_1|\omega) \\ &+ \frac{\min\{\pi'(s|L), \pi'(s|H)\}}{\rho(s_2|L)}\rho(s_2|\omega) \end{aligned}$$

and

$$\sum_{s\in S'} \frac{\max\{\pi'(s|L)-\pi'(s|H),0\}}{\rho(s_0|L)} = \frac{\sum_{s\in S'}\max\{\pi'(s|L)-\pi'(s|H),0\}}{1-\sum_{s\in S'}\min\{\pi'(s|L),\pi'(s|H)\}} = 1$$
$$\sum_{s\in S'} \frac{\max\{\pi'(s|H)-\pi'(s|L),0\}}{\rho(s_1|H)} = \frac{\sum_{s\in S'}\max\{\pi'(s|H)-\pi'(s|L),0\}}{1-\sum_{s\in S'}\min\{\pi'(s|L),\pi'(s|H)\}} = 1$$
$$\sum_{s\in S'} \frac{\min\{\pi'(s|L),\pi'(s|H)\}}{\rho(s_2|L)} = 1.$$

Hence, π' is a garbling of ρ and we have $\rho \gtrsim_B \pi'$. Note that $\pi'' \gtrsim_B \rho$ and $\rho \gtrsim_B \pi'$ implies $\pi'' \gtrsim_B \pi'$. Therefore, $\pi''(\mu=\mu_0|H) = \pi''(\mu=\mu_0|L) \leq \sum_{s\in S'}\min\{\pi'(s|L),\pi'(s|H)\}$ is a sufficient condition for $\pi'' \gtrsim_B \pi'$.

Conversely, suppose $\pi'' \gtrsim_B \pi'$. Then, there exists probability distribution $\gamma_0, \gamma_1, \gamma_{\mu_0}$ over S' such that

$$\pi'(s|\omega) = \gamma_0(s)\pi''(\mu=0|\omega) + \gamma_1(s)\pi''(\mu=1|\omega) + \gamma_{\mu_0}(s)\pi''(\mu=\mu_0|\omega)$$

for all $s\in S'$ and $\omega\in\Omega$. Then,

$$\begin{aligned}
&\sum_{s\in S'}\min\{\pi'(s|L),\pi'(s|H)\}\\
&=\sum_{s\in S'}\min\left\{\begin{matrix}\gamma_0(s)\pi''(\mu=0|L)+\gamma_{\mu_0}(s)\pi''(\mu=\mu_0|L),\\ \gamma_1(s)\pi''(\mu=1|H)+\gamma_{\mu_0}(s)\pi''(\mu=\mu_0|H)\end{matrix}\right\}\\
&=\sum_{s\in S'}\left[\min\{\gamma_0(s)\pi''(\mu=0|L),\gamma_1(s)\pi''(\mu=1|H)\}+\gamma_{\mu_0}(s)\pi''(\mu=\mu_0|L)\right]\\
&\geq\sum_{s\in S'}\gamma_{\mu_0}(s)\pi''(\mu=\mu_0|L)\\
&=\pi''(\mu=\mu_0|L)\\
&=\pi''(\mu=\mu_0|H).
\end{aligned}$$

Hence, $\pi''(\mu=\mu_0|L) = \pi''(\mu=\mu_0|H) \leq \sum_{s\in S'}\min\{\pi'(s|L),\pi'(s|H)\}$ is a necessary condition for $\pi'' \gtrsim_B \pi'$. Therefore, $\pi'' \gtrsim_B \pi'$ is equivalent to $\pi''(\mu=\mu_0|L) = \pi''(\mu=\mu_0|H) \leq \sum_{s\in S'}\min\{\pi'(s|L),\pi'(s|H)\}$, or $\pi''(\mu=0|L) = \pi''(\mu=1|H) \geq 1-\sum_{s\in S'}\min\{\pi'(s|L),\pi'(s|H)\}$. By combining the first half and the second half, it can be seen that Lemma 1 holds. □

Proof of Lemma 2. Suppose that $\pi \gtrsim_B \pi'' \gtrsim_B \pi'$ and $\text{supp}(\mu'') = \{0, \mu_0, 1\}$. From Lemma 9, we conclude that $\pi'' \gtrsim_S \pi'$ holds. From Lemma 10, we also conclude that $\pi \gtrsim_S \pi''$ holds. Therefore, we have $\pi \gtrsim_S \pi'$. □

Proof of Proposition 3. We know that there exists π'' such that $\text{supp}(\pi'') = \{0, \mu_0, 1\}$ and $\pi \gtrsim_B \pi'' \gtrsim_B \pi'$ by Lemma 1. Then, Lemma 2 shows $\pi \gtrsim_S \pi'$. □

A.4 Proof of Proposition 5

Proof. Without loss of generality, assume that $x > \mu_0$, $\text{supp}(\pi) = \{s_0, s_1, s_2\}$ and $\pi(s_0|H) = 0$, $\pi(s_1|H) = 1 - \varepsilon$, $\pi(s_2|H) = \varepsilon$, $\pi(s_0|L) = 1 - \delta$, $\pi(s_1|L) = 0$, and $\pi(s_2|L) = \delta$, where ε and δ satisfy the condition that $x = \frac{\mu_0\varepsilon}{\mu_0\varepsilon + (1-\mu_0)\delta}$. We divide decision problem $\mathcal{D}$ into three cases and construct the following equilibrium $\boldsymbol{\sigma}^*$ under $(\mathcal{D}, \pi)$.

Case (i): $B^{-1}(0) \cap B^{-1}(1) \neq \emptyset$. Fix $a^* \in B^{-1}(0) \cap B^{-1}(1)$. In this case, all agents choose a^* regardless of private signal and action histories.

Case (ii): $B^{-1}(1) \cap B^{-1}(x) = \emptyset$, $B^{-1}(0) = B^{-1}(x) = \{a_0\}$ for some $a_0 \in A$. Fix any $a_1 \in B^{-1}(1)$. Agent 1 chooses a_0 if he receives s_0 or s_2 and chooses a_1 otherwise. For $i \geq 2$, agent i chooses a_0 if she receives s_0, or receives s_2 and all previous agent takes a_0. Otherwise, i chooses a_1.

Case (iii): Otherwise. First, fix $a_0 \in B^{-1}(0)$ such that for all $z \in [x, 1]$, $B^{-1}(z) \neq \{a_0\}$. (Such a_0 must exists by Lemma 4.) In this case, agent 1 chooses action a_0 if he receives s_0, chooses action from $B^{-1}(1)$ if he receives s_1, and chooses action from $B^{-1}(x)$ if he receives s_2. For $i \geq 2$, agent i chooses action a_0 if she receives s_0 or at least one agent before i has taken a_0, chooses action from $B^{-1}(1)$ if she receives s_1, and chooses action from $B^{-1}(\frac{x^i}{x^i + (\frac{\mu_0}{1-\mu_0})^{i-1}(1-x)^i}) \backslash \{a_0\}$ if she receives s_2 and no one before i has taken action a_0 or action from $B^{-1}(1)$. Otherwise, she chooses the same action as agent $i - 1$.

In *Case (i)*, it is always optimal to take a^* regardless of the posterior belief. Hence, this strategy $\boldsymbol{\sigma}^*$ is an equilibrium and we have $V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \overline{V}_i^{\mathcal{D}}(\pi)$. In *Case (iii)*, action a_0 is taken if someone has received the signal s_0 in the past, action from $B^{-1}(1)$ is taken if someone has received the signal s_1 in the past, and action from $B^{-1}(\frac{x^i}{x^i+(\frac{\mu_0}{1-\mu_0})^{i-1}(1-x)^i})$ or an action that yields the same expected payoff is taken when everyone has received s_2 in the past. Therefore, we have $V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \overline{V}_i^{\mathcal{D}}(\pi)$. Hence, $\boldsymbol{\sigma}^*$ is an equilibrium. Then, in *case (i)* and *case (iii)*, by the same argument as Lemma 9,

$$V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*) = \overline{V}_i^{\mathcal{D}}(\pi) \geq \overline{V}_i^{\mathcal{D}}(\pi') \geq V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**}),$$

for any equilibrium $\boldsymbol{\sigma}^{**}$ under $(\mathcal{D}, \pi')$.

The only case left is *Case (ii)*. In *Case (ii)*, from Lemma 5,

$$\begin{aligned} V_i^{\mathcal{D}}(\pi', \boldsymbol{\sigma}^{**}) =& \mu_0[(1-(1-\pi'(s_1|H))^i)u(a_1,H)+ \\ & (1-\pi'(s_1|H))^i u(a_0,H)] + (1-\mu_0)u(a_0,L), \end{aligned}$$

for any equilibrium $\boldsymbol{\sigma}^{**}$ under $(\mathcal{D}, \pi')$. Since $\pi'(s_1|H) \leq 1-\varepsilon$ (by $\pi \gtrsim_B \pi'$) and $u(a_1,H) > u(a_0,H)$, it follows that

$$\begin{aligned} & \mu_0[(1-(1-\pi'(s_1|H))^i)u(a_1,H) + (1-\pi'(s_1|H))^i u(a_0,H)] + (1-\mu_0)u(a_0,L) \\ &= \mu_0 u(a_1,H) - \mu_0(1-\pi'(s_1|H))^i[u(a_1,H) - u(a_0,H)] + (1-\mu_0)u(a_0,L) \\ &\leq \mu_0 u(a_1,H) - \mu_0\varepsilon^i[u(a_1,H) - u(a_0,H)] + (1-\mu_0)u(a_0,L) \\ &= V_i^{\mathcal{D}}(\pi, \boldsymbol{\sigma}^*), \end{aligned}$$

where $\boldsymbol{\sigma}^*$ is an equilibrium described above. Therefore, $\pi \gtrsim_W \pi'$. □

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