# Singularity-Based Consistent QML Estimation of Multiple Breakpoints in High-Dimensional Factor Models

Jiangtao Duan<sup>1</sup>, Jushan Bai<sup>2</sup>, Xu Han<sup>3</sup>

<sup>1</sup>Xidian University, <sup>2</sup>Columbia University and <sup>3</sup>City University of Hong Kong

#### Abstract:

This paper investigates the estimation of high-dimensional factor models in which factor loadings undergo an unknown number of structural changes over time. Given that a model with multiple changes in factor loadings can be observationally indistinguishable from one with constant loadings but varying factor variances, this reduces the high-dimensional structural change problem to a lower-dimensional one. Due to the presence of multiple breakpoints, the factor space may expand, potentially causing the pseudo factor covariance matrix within some regimes to be singular. We define two types of breakpoints: a singular change, where the number of factors in the combined regime exceeds the minimum number of factors in the two separate regimes, and a rotational change, where the number of factors in the combined regime equals that in each separate regime. Under a singular change, we derive the properties of the small eigenvalues and establish the consistency of the QML estimators. Under a rotational change, unlike in the single-breakpoint case, the pseudo factor covariance matrix within each regime can be either full rank or singular, yet the QML estimation error for the breakpoints remains stably bounded. We further propose an information criterion (IC) to estimate the number of breakpoints and show that, with probability approaching one, it accurately identifies the true number of structural changes. Monte Carlo simulations confirm strong finite-sample performance. Finally, we apply our method to the FRED-MD dataset, identifying five structural breaks in factor loadings between 1959 and 2024.

Key words and phrases: Factor model, Multiple breakpoints, Consistency, In-

formation criterion.

<sup>&</sup>lt;sup>1</sup>School of Mathematics and Statistics, Xidian University. E-mail: duanjiangtao@xidian.edu.cn

<sup>&</sup>lt;sup>2</sup>Department of Economics, Columbia University. E-mail: jb3064@columbia.edu

<sup>&</sup>lt;sup>3</sup>Department of Economics and Finance, City University of Hong Kong. E-mail: xuhan25@cityu.edu.hk

## 1 Introduction

Factor models have become increasingly popular in the analysis of economic and financial data with large cross-section and time dimensions. In practice, a few common factors usually provide a good summary of the common driving force for a large number of variables, so they can preserve the key information of the major shocks in an economic system while reducing dimensionality. Factor models are widely applied in various fields of economics, such as policy evaluation (Stock and Watson, 2016), macroeconomic modelling (Boivin and Giannoni, 2006), and asset pricing (Giglio and Xiu, 2021). While factor models are highly useful, practitioners should be cautious about the potential structural changes, which are likely to arise in a data-rich environment. Ignoring the underlying changes in factor models may lead to misleading results.

In this paper, we propose a quasi-maximum likelihood method for estimating breakpoints in high-dimensional factor models with multiple structural changes by introducing a unified framework that classifies breakpoints as singular or rotational changes, with asymptotic theory establishing eigenvalue convergence rate and estimator consistency for singular changes and bounded estimation error for rotational changes. Moreover, we develop an information criterion that consistently identifies the true number of breakpoints with probability approaching one, addressing the practical challenge of unknown breakpoint numbers. Duan, Bai, and Han (2021, hereafter DBH) demonstrated that when a single structural break occurs in the factor loading matrix, the resulting factor model becomes observationally equivalent to one with time-invariant loadings and possibly more pseudo-factors. This expansion of the factor space induces singularity in the covariance matrices of the pseudo-factors before and/or after the break. The singularity is the key condition to generate the consistency of the proposed QML estimator. In practical scenarios, particularly when dealing with a substantial time dimension T, multiple breakpoints may arise, thereby rendering the DBH approach inapplicable.

The concept of multiple breakpoints extends far beyond the framework of a single breakpoint, introducing a fundamentally different scenario that faces several significant challenges: First, while singularity is a key criterion for verifying the consistency of estimated breakpoints in a single-breakpoint setting, it is not the sole indicator in a multiple-breakpoint context, as rotational changes can also result in singular covariance matrices. For example, in a multiple-breakpoint scenario, adjacent regimes may simultaneously exhibit both rotational and singular characteristics-meaning that although these regimes share the same number of factors, their loading matrices differ only by a rotational transformation, and the factor dimension within these regimes is lower than the overall number of pseudo-factors estimated from the entire sample. Such a configuration is naturally excluded in a single-breakpoint framework via dimensionality reduction during the factor estimation; however, in multiplebreakpoint models, additional structural changes expand the overall factor space, making such exclusion infeasible. Consequently, this complexity complicates the task of determining whether the estimated breakpoints are consistent or merely bounded. Second, establishing that the information criterion consistently identifies the true number of breaks with probability approaching one (w.p.a.1) poses significant technical challenges. This difficulty arises from the need to characterize the divergence rates of the objective function under both overfitting and underestimating breakpoints. In a multi-breakpoint context, these divergence rates may vary depending on the specific type of breakpoint, further complicating the theoretical analysis.

To address these challenges, we extend the QML estimator for the singular break case to the multiple breaks case, designed to simultaneously address more complex data structures. This paper makes the following contributions to the literature. First, we propose a unified theoretical framework that classifies structural breaks into two distinct types: singular changes and rotational changes. Furthermore, we establish a necessary and sufficient conditions, along with their equivalent representations, to precisely distinguish between these two types of structural breaks.

Second, we establish the convergence rate for the distance between the eigenvalues of the estimated factor covariance matrix and those of the sub-regime covariance matrix when the true factor covariance matrix in that regime is singular. This conclusion plays a crucial role in the subsequent theoretical analysis, including the consistency and boundedness of the QML estimators, as well as the consistency of the information criterion.

Third, we establish the consistency of the QML estimator in factor model with multiple breaks. Specifically, if the number of factors within the combined regime  $[k_{j-1}^0, k_{j+1}^0]$  exceeds the number of factors in the regimes  $[k_{j-1}^0, k_j^0]$  or  $[k_j^0, k_{j+1}^0]$ , or both of them, then the QML estimator  $\hat{k}_j$  is consistent. Conversely, if the number of factors within  $[k_{j-1}^0, k_j^0]$ ,  $[k_j^0, k_{j+1}^0]$ , and  $[k_{j-1}^0, k_{j+1}^0]$  are equal, then the QML estimator  $\hat{k}_j$  is stochastically bounded. In this scenario, the loading matrix undergoes solely a rotational change between the regimes  $[k_{j-1}^0, k_j^0]$  and  $[k_j^0, k_{j+1}^0]$ . Notably, we do not require the number of factors within the regimes  $[k_{j-1}^0, k_j^0]$  and  $[k_j^0, k_{j+1}^0]$  to equal the number of factors in the full sample. In contrast, in a single-breakpoint scenario, this equality is necessary.

Fourth, we introduce an effective information criterion to determine the number of breakpoints, which achieves a balanced trade-off between the loss function and the penalty term. We demonstrate that the difference in the loss function is  $O_p(1)$  in the case of overfitting and  $O_p(T)$  when the number of breakpoints is underestimated. As a result, the penalty function must be carefully chosen to balance these different rates effectively. Finally, we provide theoretical proof that this information criterion can consistently select the correct number of breakpoints.

Our paper is also related to, but substantially different from, other studies estimating structural changes in factor models. Earlier literature establishes the consistency of the estimated break fraction (i.e., the break date  $k_0$  divided by the sample size T) (e.g., Chen, 2015 and Cheng et al. 2016). Barigozzi et al. (2018) propose a wavelet-based method that consistently estimates both the number and locations of structural breaks in the common and idiosyncratic components. By comparing the second moments of factors estimated via fullsample principal components, Baltagi et al. (2017, 2021) developed an estimator that permits changes in the number of factors but produces estimation errors that are only stochastically bounded, not consistent with the true breakpoint positions. Under a small break setup, Bai, Han, and Shi (2020) obtain a much stronger result that the LS estimator is consistent for the break point, i.e., the estimated break point is equal to the true one with a probability approaching one in large samples. Their method involves performing PCA across all potential splitting points for covariance matrices before and after the split. By contrast, our QML methodology conducts PCA just once on the entire dataset, resulting in greater computational efficiency, a difference that becomes particularly pronounced when dealing with extensive data sets. Ma and Su (2018) propose a fused Lasso estimator that is consistent for multiple break points in factor models. Compared to the approaches of Ma and Su (2018) and Bai et al. (2020), who assume that the number of factors remains constant after the break, our method offers greater generality.

Lastly, we perform a series of Monte Carlo simulations to assess the finite sample properties of our procedure. The results demonstrate that our method can accurately estimate the break dates, even in the presence of small sample sizes, and that our information criterion consistently identifies the correct number of breaks. We further apply our approach to the FRED-MD dataset (McCracken and Ng, 2016), detecting five breakpoints over the period from January 1959 to July 2024.

The rest of this paper is organized as follows. Section 2 introduces the factor model with multiple breaks on the factor loading matrix and describes the QML estimators for break dates. Section 3 presents the assumptions made for this model. Section 4 studies the asymptotic theory. Section 5 investigates the finite-sample properties of the QML estimators through simulations. Section 6 provides an empirical study. Section 7 concludes.

The following notations will be used throughout the paper. Let  $\rho_i(\mathbb{B})$  denote the *i*-th eigenvalue of an  $n \times n$  symmetric matrix  $\mathbb{B}$ , and  $\rho_1(\mathbb{B}) \ge \rho_2(\mathbb{B}) \ge \cdots \ge \rho_n(\mathbb{B})$ . For an  $m \times n$  real matrix  $\mathbb{A}$ , we denote its Frobenius norm as  $\|\mathbb{A}\| = [tr(\mathbb{A}\mathbb{A}')]^{1/2}$ , and its adjoint matrix as  $\mathbb{A}^{\#}$  when m = n. Let  $\operatorname{Proj}(\mathbb{A}|\mathbb{Z})$  denote the projection of matrix  $\mathbb{A}$  onto the columns of matrix  $\mathbb{Z}$ . For a real number x, [x] represents the integer part of x.

#### 2 Model and estimator

Let us consider the following factor model with  $m_0$  common breaks at  $k_1^0(T) < k_2^0(T) < \cdots < k_{m_0}^0(T)$  in the factor loadings for  $i = 1, \cdots, N$ :

$$x_{it} = \begin{cases} \lambda_{i,1}f_t + e_{it} & for \ t = 1, 2, \cdots, k_1^0(T) \\ \dots \\ \lambda_{i,m_0+1}f_t + e_{it} & for \ t = k_{m_0}^0(T) + 1, \cdots, T, \end{cases}$$
(1)

where  $f_t$  is an r-dimensional vector of unobserved common factors; r is the number of pseudo-factors in the entire sample;  $k_j^0(T), j = 1, \dots, m_0$  are unknown break dates and  $\min_{\{j=0,1,\dots,m_0\}} k_{j+1}^0(T) - k_j^0(T) \ge cT$  for constant c, with  $k_0^0(T) = 0$  and  $k_{m_0+1}^0(T) = T$ ;  $\lambda_{i,j}, j = 1, \dots, m_0 + 1$  are factor loadings in different time periods; and  $e_{it}$  is the error term allowed to have serial and cross-sectional dependence as well as heteroskedasticity, and both N and T tend to infinity.  $\tau_j^0 = k_j^0(T)/T \in (0, 1), j = 1, \dots, m_0$  are break fractions and are fixed constants. For notational simplicity, hereinafter, we suppress the dependence of  $k_j^0$  on T. Note that the dimension of  $f_t$  is the same as that of the pseudo-factors (to be defined soon) instead of the original underlying factors.

In vector form, model (1) can be expressed as

$$x_{t} = \begin{cases} \Lambda_{1}f_{t} + e_{t} & for \ t = 1, 2, \cdots, k_{1}^{0} \\ \dots \\ \Lambda_{m_{0}+1}f_{t} + e_{t} & for \ t = k_{m_{0}}^{0} + 1, \cdots, T, \end{cases}$$
(2)

where  $x_t = [x_{1,t}, \cdots, x_{N,t}]'$ ,  $e_t = [e_{1,t}, \cdots, e_{N,t}]'$ ,  $\Lambda_j = [\lambda_{1,j}, \cdots, \lambda_{N,j}]'$ ,  $j = 1, \cdots, m_0 + 1$ . For any  $j = 1, \cdots, m_0 + 1$ , we define

$$X_j = [x_{k_{j-1}^0+1}, \cdots, x_{k_j^0}]', \quad F_j = [f_{k_{j-1}^0+1}, \cdots, f_{k_j^0}]', \quad \boldsymbol{e}_j = [e_{k_{j-1}^0+1}, \cdots, e_{k_j^0}]'$$

We rewrite (2) using the following matrix representation:

$$X = \begin{bmatrix} X_1 \\ \cdots \\ X_{m_0+1} \end{bmatrix} = \begin{bmatrix} F_1\Lambda'_1 \\ \cdots \\ F_{m_0+1}\Lambda'_{m_0+1} \end{bmatrix} + \begin{bmatrix} e_1 \\ \cdots \\ e_{m_0+1} \end{bmatrix} = \begin{bmatrix} F_1(\Lambda B_1)' \\ \cdots \\ F_{m_0+1}(\Lambda B_{m_0+1})' \end{bmatrix} + \begin{bmatrix} e_1 \\ \cdots \\ e_{m_0+1} \end{bmatrix},$$
$$= \underbrace{\begin{bmatrix} F_1B'_1 \\ \cdots \\ F_{m_0+1}B'_{m_0+1} \end{bmatrix}}_{G} \Lambda' + \begin{bmatrix} e_1 \\ \cdots \\ e_{m_0+1} \end{bmatrix},$$
$$= G\Lambda' + E, \qquad (3)$$

where  $F_j$ ,  $j = 1, \dots, m_0 + 1$  have dimensions  $(k_j^0 - k_{j-1}^0) \times r$  and  $\Lambda$  is an  $N \times r$  matrix with full column rank. The loadings are modeled as  $\Lambda_j = \Lambda B_j$ ,  $j = 1, \dots, m_0 + 1$ , where  $B_j$  is an  $r \times r$ matrix. Each  $\Lambda_j$  has dimension  $N \times r$ . In this model,  $r_j = rank(B_j)(\leq r)$  denote the numbers of original factors in regime  $[k_{j-1}^0, k_j^0]$ . In (3), we define G as the pseudo-factors because the final expression in (3) provides an observationally equivalent representation, maintaining the structure of the loadings matrix  $\Lambda$  unchanged. More precisely, when the break is omitted during the estimation, the factors derived from a full-sample PCA correspond to the pseudofactors G presented in (3). It is well-established that the presence of breaks can expand the factor space; consequently, for  $j = 1, \dots, m_{0+1}$ , we have  $r_j \leq r$ , where rank(G) = r. This representation allows for changes in factor loadings and the number of factors. When there are no breaks in all factor loadings,  $B_j = I_r$  for  $j = 1, \dots, m_0 + 1$  in model (3).

The representation in (3) is convenient for theoretical analysis because one can control break types by setting  $B_j$  for  $j = 1, \dots, m_0 + 1$ . To further illustrate the generality of model (3), we present two types of breakpoint structures within the combined regime  $[k_{j-1}^0, k_{j+1}^0]$  by varying the ranks of  $B_j$  and  $B_{j+1}$ . Define  $r_{j,j+1} = \operatorname{rank}([B_jF'_j, B_{j+1}F'_{j+1}]') = \operatorname{rank}([B_j, B_{j+1}])$ .

 $^{1}$ Since

$$\begin{aligned} r_{j,j+1} &= \operatorname{rank}([B_jF'_j, B_{j+1}F'_{j+1}]') &= \operatorname{rank}([B_jF'_j, B_{j+1}F'_{j+1}][B_jF'_j, B_{j+1}F'_{j+1}]'/(k^0_{j+1} - k^0_{j-1})) \\ &= \operatorname{rank}\left([B_j, B_{j+1}]diag(F'_jF_j/(k^0_{j+1} - k^0_{j-1}), F'_{j+1}F_{j+1}/(k^0_{j+1} - k^0_{j-1}))[B_j, B_{j+1}]'\right) \\ &= \operatorname{rank}\left([B_j, B_{j+1}]\right), \end{aligned}$$

Next, we define two types of breakpoints based on the relationship between  $r_{j,j+1}$  and  $\min\{r_j, r_{j+1}\}.$ 

**Type 1**.  $r_{j,j+1} > \min\{r_j, r_{j+1}\}$ . In this case, the number of pseudo factors in the combined regime  $[k_{j-1}^0, k_{j+1}^0]$  is larger than the number of true factors <sup>2</sup> in either or both of the two regimes  $[k_{j-1}^0, k_j^0]$  and  $[k_j^0, k_{j+1}^0]$ , we refer to this as a singular change. If  $r_{j,j+1} > \max\{r_j, r_{j+1}\}$ , this implies that the number of true factors within regimes  $[k_{j-1}^0, k_j^0]$ and  $[k_j^0, k_{j+1}^0]$  is strictly less than the number of pseudo-factors within the combined regime  $[k_{j-1}^0, k_{j+1}^0]$  (i.e.,  $r_j < r_{j,j+1}$  and  $r_{j+1} < r_{j,j+1}$ ). As a result, the dimension of the factor loadings space within the combined regime  $[k_{j-1}^0, k_{j+1}^0]$  increases due to the structural breakpoint. For example, when the difference in factor loadings before and after the breakpoint  $k_j^0$  (i.e.,  $\Lambda_j - \Lambda_{j+1}$ ) is linearly independent of the factor loadings after the breakpoint  $(\Lambda_{j+1})$ , we have  $r_j < r_{j,j+1}$  and  $r_{j+1} < r_{j,j+1}$ . When the factor loadings before and after the breakpoint  $k_j^0$ are linearly independent, then  $r_{j,j+1} = r_j + r_{j+1}$ . If  $r_{j,j+1} = \max\{r_j, r_{j+1}\}$ , this implies that new factors may emerge or existing factors may disappear from regime  $[k_{j-1}^0, k_j^0]$  to  $[k_j^0, k_{j+1}^0]$ . For instance, if  $r_{j,j+1} = r_{j+1}$  and  $r_j = \operatorname{rank}(B_j) < r_{j,j+1}$ , it indicates that some new factors will appear after the breakpoint  $k_i^0$ .

**Type 2**.  $r_{j,j+1} = \min\{r_j, r_{j+1}\}$ . In this case, the number of pseudo factors in the combined regime  $[k_{j-1}^0, k_{j+1}^0]$  is the same as the number of factors in each regime, and we refer to this as a rotational change. Specifically, there exists a nonsingular  $r \times r$  matrix **R** such that

$$B_j \mathbf{R} = B_{j+1},$$

then

$$\begin{bmatrix} F_j B'_j \\ F_{j+1} B'_{j+1} \end{bmatrix} = \begin{bmatrix} F_j B'_j \\ F_{j+1} \mathbf{R}' B'_j \end{bmatrix} = \begin{bmatrix} F_j \\ F_{j+1} \mathbf{R}' \end{bmatrix} B'_j, \tag{4}$$

which satisfies  $r_{j,j+1} = r_j = r_{j+1} = rank(B_j)$ . When there is one break  $(m_0 = 1)$  and  $B_1$ is nonsingular, it is analogous to the rotational change for single break setup. For example, if  $B_2 = 2B_1$  and  $m_0 = 1$ , it indicates that the factors after the breakpoint have twice standard deviation as those before the breakpoint, it must follow that  $r = r_{1,2} = r_1 = r_2$  for

it follows that  $r_{j,j+1} = \operatorname{rank}([B_j F'_j, B_{j+1} F'_{j+1}]') = \operatorname{rank}([B_j, B_{j+1}])$ . <sup>2</sup>The term "true factor" here refers to the number of factors within a regime that has no breakpoints.

 $m_0 = 1$ . Such a breakpoint suggests that a model exhibiting a rotational change in the factor covariance matrix, with unchanged loadings, can be equivalently regarded as a Type 2 change. When  $B_j$  is singular, the rotational change is also associated with a singular, indicating a breakpoint that exhibits both rotational and singular characteristics  $(r > r_{j,j+1} = r_j = r_{j+1})$ , as demonstrated in DGP 1.E of the simulation section. Therefore, the multiple breakpoint setting in model (3) includes a broader range of potential breakpoint configurations, making it significantly more complex than the single breakpoint setting.

**Remark 1** Singularity is not the unique feature of a singular change. Although type 1 is labeled as a "singular change" and type 2 as a "rotational change", singularity can occur in both cases. Thus, singularity alone is insufficient to differentiate between the two types of changes. To understand the origin of singularity, consider equation (4): if  $B_j$  is singular, the subsample associated with the rotational type may not have full column rank. When analyzing the data within the combined regime  $[k_{j-1}^0, k_{j+1}^0]$ , the pseudo-factors are full rank, with the number of factors denoted as  $r_{j,j+1}$ . However, in the context of the full sample, the pseudo-factors within this combined regime are not full rank. This singularity is not directly attributable to the *j*-th breakpoint itself but rather results from the expansion of the factor loadings space caused by other breakpoints, which renders the pseudo-factors in  $[k_{j-1}^0, k_{j+1}^0]$ not full rank. Consequently, the j-th breakpoint is still classified as a rotational type. This highlights that singularity is not introduced solely by the j-th breakpoint but rather by the cumulative effect of structural changes at multiple locations. Therefore, relying solely on the ranks of  $B_j$  and  $B_{j+1}$  is insufficient to distinguish between type 1 and type 2 changes at *j*-th breakpoint, as both types can lead to singularity in  $B_j$  and  $B_{j+1}$ . Situations involving both singular and rotational changes occur only when multiple breakpoints are present; such scenarios do not arise in single-breakpoint cases. To determine whether a breakpoint is of type 2, we establish that equation (4) provides a necessary and sufficient condition for classification as type 2, as demonstrated in Proposition 1.

**Proposition 1** There exists a nonsingular **R** such that  $B_j \mathbf{R} = B_{j+1} \Leftrightarrow r_{j,j+1} = \min(r_j, r_{j+1})$ .

**Remark 2** Proposition 1 can equivalently be written as: There does not exists a nonsingular

**R** such that  $B_j \mathbf{R} = B_{j+1} \Leftrightarrow r_{j,j+1} > \min(r_j, r_{j+1})$ , which is corresponding to a type 1 change. To further understand Proposition 1, let's consider a simple example. Let

$$B_{1} = \begin{bmatrix} 1, 0, 0 \\ 0, 0, 0 \\ 0, 0, 0 \end{bmatrix}, B_{2} = \begin{bmatrix} 1, 1, 0 \\ 0, 0, 0 \\ 0, 0, 0 \end{bmatrix}, \tilde{B}_{2} = \begin{bmatrix} 1, 0, 0 \\ 1, 0, 0 \\ 0, 0, 0 \end{bmatrix}, B_{3} = I_{3},$$

then  $rank(B_1) = rank(B_2) = rank(\tilde{B}_2) = 1$ . But  $rank([B_1, B_2]) = 1$  and  $rank([B_1, \tilde{B}_2]) = 2 > rank(B_1)$ . For  $B_1$ , we can find a nonsingular  $r \times r$  matrix **R** such that

$$B_1 \times \mathbf{R} = \begin{bmatrix} 1, 0, 0 \\ 0, 0, 0 \\ 0, 0, 0 \end{bmatrix} \times \begin{bmatrix} 1, 1, 0 \\ 0, 1, 0 \\ 0, 0, 1 \end{bmatrix} = \begin{bmatrix} 1, 1, 0 \\ 0, 0, 0 \\ 0, 0, 0 \end{bmatrix} = B_2$$

But such nonsingular  $\mathbf{R}$  does not exist for  $\tilde{B}_2$  because the second row of the product  $B_1R$  is always zeros, which is not equal to that of  $\tilde{B}_2$ , i.e., for any  $\mathbf{R}$ ,

$$B_1 \times \mathbf{R} \neq \begin{bmatrix} 1, \ 0, \ 0\\ 1, \ 0, \ 0\\ 0, \ 0, \ 0 \end{bmatrix} = \tilde{B}_2.$$

Thus, in this example, for  $B_2$ , since the number of pseudo factors in the combined regime is equal to the number of factors in regimes 1 and 2 ( $r_{1,2} = r_1 = r_2 < r$ ), the first breakpoint is classified as type 2, i.e., a rotational change. In contrast, for  $\tilde{B}_2$ , the number of pseudo factors in the combined regime exceeds that in each regime ( $r_{1,2} > \min\{r_1, r_2\}$ ), indicating that this breakpoint is classified as type 1, i.e., a singular change.

Analyzing the model with multiple changes is more complex than the model with single change. Because in a single breakpoint model, only one type of breakpoint exist in the model, while in the multiple changes case, both types of breakpoints may exist. This greatly increases the complexity of analysis and estimation. Therefore, it is necessary to analyze the convergence rate of each type of breakpoint.

In this section, we consider using the QML (quasi-maximum likelihood) method to estimate the break dates for model (3), assuming the number of breaks is known. We also propose an information criterion to determine the number of breaks in section 4. Let  $\hat{G} = (\hat{g}_1, ..., \hat{g}_T)'$  denote the full-sample PCA estimator for G subject to the normalization condition that  $G'G/T = I_r$  and  $\Lambda'\Lambda$  being diagonal. The number of pseudo factors can be estimated using Bai and Ng (2002). For a given set of partition  $(k_1, \cdots, k_{m_0})$ , the QML objective function is expressed as:

$$U_{NT}(k_1, \cdots, k_{m_0}) = \sum_{\ell=1}^{m_0+1} (k_\ell - k_{\ell-1}) \log |\hat{\Sigma}(k_{\ell-1}, k_\ell)|,$$

where

$$\hat{\Sigma}(k_{\ell-1},k_{\ell}) = \frac{1}{k_{\ell}-k_{\ell-1}} \sum_{t=k_{\ell-1}+1}^{k_{\ell}} \hat{g}_t \hat{g}_t',$$

with  $k_0 = 0$  and  $k_{m_0+1} = T$ . The QML estimator of the break points for model (3) is defined as follows:

$$(\hat{k}_1, \cdots, \hat{k}_{m_0}) = \arg\min_{(k_1, \cdots, k_{m_0}) \in K_\tau} U_{NT}(k_1, \cdots, k_{m_0}),$$
 (5)

where  $K_{\tau} = \{(k_1, \cdots, k_{m_0}) : k_j - k_{j-1} \ge T\tau, 1 < k_j < T, j = 1, \cdots, m_0 + 1\}$  with  $\tau \in (0, 1)$ .

Similar to the argument of DBH and Baltagi, Kao, and Wang (2017, 2021, BKW hereafter), the second moment of  $\hat{g}_t$  shared the same change point as that of  $g_t$ . Therefore, we can obtain the QML break point estimators  $\hat{k}_j$ ,  $j = 1, \dots, m_0$  using (5).

**Remark 3** Bai and Perron (2003) introduced a dynamic programming algorithm, providing an efficient method for comparing possible combinations to minimize the global sum of squared residuals, requiring at most least-squares operations of order  $O(T^2)$  for any number of breaks. In this paper, while estimating the globally optimal breakpoints, we apply this approach to efficiently identify the global minimizers of the QML objective function.

### 3 Assumptions

The assumptions are as follows:

Assumption 1 (i)  $E ||f_t||^4 < M < \infty$ ,  $E(f_t f'_t) = \Sigma_F$ , where  $\Sigma_F$  is positive definite, and  $\frac{1}{k_j^0 - k_{j-1}^0} \sum_{t=k_{j-1}^{0}+1}^{k_j^0} f_t f'_t \xrightarrow{p} \Sigma_F$  for  $j = 1, \cdots, m_0 + 1$ ;

(ii) There exists d > 0 such that  $\|\Delta_j\| \ge d > 0$  for  $j = 2, \dots, m_0 + 1$ , where  $\Delta_j = B_j \Sigma_F B'_j - B_{j-1} \Sigma_F B'_{j-1}$  and  $B_j$  are  $r \times r$  matrices.

**Assumption 2**  $\|\lambda_{i,j}\| \leq \overline{\lambda} < \infty$  for  $j = 1, \dots, m_0 + 1$ ,  $i = 1, \dots, N$ ,  $\left\|\frac{1}{N}\Lambda'\Lambda - \Sigma_{\Lambda}\right\| \to 0$  for some  $r \times r$  positive definite matrix  $\Sigma_{\Lambda}$ .

**Assumption 3** There exists a positive constant  $M < \infty$  such that

- (i)  $E(e_{it}) = 0$  and  $E|e_{it}|^8 \le M$  for all  $i = 1, \dots, N$  and  $t = 1, \dots, T$ ;
- (*ii*)  $E(e'_{s}e_{t}/N) = E(N^{-1}\sum_{i=1}^{N}e_{is}e_{it}) = \gamma_{N}(s,t) \text{ and } \sum_{s=1}^{T}|\gamma_{N}(s,t)| \le M \text{ for every } t \le T;$
- (iii)  $E(e_{it}e_{jt}) = \tau_{ij,t}$  with  $|\tau_{ij,t}| < \tau_{ij}$  for some  $\tau_{ij}$  and for all  $t = 1, \dots, T$  and  $\sum_{j=1}^{N} |\tau_{ij}| \le M$ for every  $i \le N$ ;
- $(iv) E(e_{it}e_{js}) = \tau_{ij,ts},$

$$\frac{1}{NT} \sum_{i,j,t,s} |\tau_{ij,ts}| \le M;$$

- (v) For every (s,t),  $E \left| N^{-1/2} \sum_{i=1}^{N} (e_{is} e_{it} E[e_{is} e_{it}]) \right|^4 \le M;$
- (vi) For every (i, j),  $E \left| (k_j^0 k_{j-1}^0)^{-1/2} \sum_{t=k_{j-1}^0+1}^{k_j^0} (e_{it}e_{jt} E[e_{it}e_{jt}]) \right|^4 \le M, \ j = 1, \cdots, m_0 + 1.$

**Assumption 4** There exists a positive constant  $M < \infty$  such that

$$E\left(\frac{1}{N}\sum_{i=1}^{N}\left\|\frac{1}{\sqrt{k_{j}^{0}-k_{j-1}^{0}}}\sum_{t=k_{j-1}^{0}+1}^{k_{j}^{0}}f_{t}e_{it}\right\|^{2}\right) \leq M \text{ for } j=1,\cdots,m_{0}+1.$$

**Assumption 5** The eigenvalues of  $\Sigma_G \Sigma_{\Lambda}$  are distinct.

Assumption 6 Define  $\epsilon_t = f_t f'_t - \Sigma_F$ . The Hájek-Rényi inequality applies to the processes  $\{\epsilon_t, t = k_{j-1}^0 + 1, \cdots, k_j^0\}$  and  $\{\epsilon_t, t = k_j^0, \cdots, k_{j-1}^0 + 1\}$  for  $j = 1, \cdots, m_0 + 1$ .

Assumption 7 There exists an  $M < \infty$  such that

(i) for each 
$$h = 1, \cdots, T$$
,  

$$E\left(\max_{k>s} \frac{1}{k-s} \sum_{t=s+1}^{k} \left| \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left[ e_{ih} e_{it} - E(e_{ih} e_{it}) \right] \right|^{2} \right) \leq M;$$
(ii)

$$E\left(\max_{k>s}\frac{1}{k-s}\sum_{t=s+1}^{k}\left\|\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\lambda_{i}e_{it}\right\|^{2}\right) \leq M.$$

**Assumption 8** There exists an  $M < \infty$  such that for all values of N and T,

(i)

$$E\left(\max_{k>s}\frac{1}{k-s}\sum_{t=s+1}^{k}\left\|\frac{1}{\sqrt{NT}}\sum_{h=1}^{T}\sum_{i=1}^{N}f_{s}[e_{ih}e_{it}-E(e_{ih}e_{it})]\right\|^{2}\right) \leq M.$$

(ii)

(iii)

$$\lim \sup_{N,k>s} \left\| \frac{1}{\sqrt{N(k-s)\log\log(N(k-s))}} \sum_{t=s+1}^k \sum_{i=1}^N \lambda_i f'_t e_{it} \right\|^2 \le M.$$

$$\lim \sup_{N,k>s} \left\| \frac{1}{N\sqrt{k-s}\sqrt{\log\log N^2(k-s)}} \sum_{t=s+1}^k \sum_{i=1}^N \sum_{h=1}^N \lambda_i \lambda_h' (e_{it}e_{ht} - E(e_{it}e_{ht})) \right\| \le M$$

Assumptions 1-5 are standard in the factor model literature. Assumptions 1 (i) is similar to Assumption A in Bai (2003) and (ii) imposes restrictions on  $B_j$  and  $B_{j-1}$  to ensure the identification of the break point. Assumptions 1 (ii) rules out the case that  $B_j = -B_{j-1}$  since the objective function (5) can not identify the difference between  $\hat{\Sigma}_{j-1}$  and  $\hat{\Sigma}_j$ . Assumptions 2 is similar to Assumption B of Bai (2003). Assumption 3 allows for weakly correlated error terms in both time and cross-sectional dimensions. Assumption 4 means that the factors and idiosyncratic errors are allowed to be weakly dependent within each regime. Assumption 5 corresponds to Assumption G in Bai (2003). Assumption 6 corresponds to Assumption 7 of Baltagi et al. (2021), which allows the Hájek-Rényi inequality to be applicable to the second moment process of the factors. Assumptions 7 and 8 correspond to Assumptions 7 and 8 of DBH. Assumptions 7 and 8(i) imposes further constraints on the idiosyncratic error, and Assumption 8(ii) and (iii) are Law of the Iterated Logarithm (LIL).

Assumption 9 With probability approaching one (w.p.a.1), the following inequalities hold:

$$0 < \underline{c} \le \min_{k-s \ge [\tau T]} \rho_h \left( \frac{1}{N(k-s)} \sum_{t=s+1}^k \Lambda' e_t e'_t \Lambda \right), \text{ for } h = 1, \cdots, r;$$
  
$$\rho_1 \left( \frac{1}{NT} \sum_{t=1}^T \Lambda' e_t e'_t \Lambda \right) \le \overline{c} < +\infty,$$

as  $N, T \to \infty$ , where  $\underline{c}$  and  $\overline{c}$  are some constants.

#### Assumption 10

$$\max_{k-s \ge [\tau T]} \left\| \frac{1}{\sqrt{N(k-s)}} \sum_{t=s+1}^{k} \sum_{i=1}^{N} f_t e_{it} \lambda_i' \right\| = O_p(1).$$

Assumption 11 (i)  $[B_1, \dots, B_{m_0}]$  is of full row rank.

(ii) For  $j = 1, \dots, m_0$ ,  $||B_j f_{k_j^0} - \operatorname{Proj}(B_j f_{k_j^0} | B_{j+1})|| \ge d > 0$  when  $r - r_{j+1} \ge 2$  or  $r_{j+1} = 0$ ; and  $||B_{j+1} f_{k_{j+1}^0 + 1} - \operatorname{Proj}(B_{j+1} f_{k_{j+1}^0} | B_j)|| \ge d > 0$  when  $r - r_j \ge 2$  or  $r_j = 0$ , where  $\operatorname{Proj}(\mathbb{A}|\mathbb{Z})$ denotes the projection of  $\mathbb{A}$  onto the columns of  $\mathbb{Z}$ , and d is a constant.

Assumption 9 is the same as Assumption 9 of DBH, which is an extension of Assumption F3 in Bai (2003). Assumption 10 is an extension of 8 (ii), which is similar to Assumption F2 of Bai (2003). Assumption 11 (i) implies that  $\Sigma_G$  is positive definite. Assumption 11 (ii) is similar with Assumption 11 of DBH.

## 4 Asymptotic properties of the QML estimators

In this section, we establish the asymptotic properties of the quasi-maximum likelihood (QML) estimators. In the existing literature on structural breaks within fixed-dimensional time series, traditional breakpoint estimators, such as Bai's (1997a) least-squares (LS) estimator, Bai's (2000) QML estimator, and the method proposed by Qu and Perron (2007), are shown to exhibit fraction consistency. However, for the QML estimators to achieve consistency in breakpoint estimation, the cross-sectional dimension of the time series must be large, as demonstrated in works such as Bai (2010) and Kim (2011).

Analyzing multiple structural changes is significantly more complex than examining a single change due to increased uncertainty in identifying regime boundaries. In a single-break case, where two regimes exist, the boundaries are partially known the first observation marks the start of the initial regime, while the last observation defines the end of the final regime. This partial knowledge simplifies the problem. However, with multiple breaks, hypothesized regimes may not align with the actual ones, increasing difficulty and complexity. For a single breakpoint, the search spans the entire interval [1, T]. When two breakpoints are involved, all possible pairs  $(k_1, k_2)$  must be considered, ensuring  $k_1$  and  $k_2$  are separated by at least  $\tau T$ , where  $\tau$  is typically a small positive integer. This constraint reduces computational complexity and eases theoretical derivations. Since  $\tau$  can be very small, hypothesized breakpoints  $k_1, \ldots, k_m$  are allowed within a single regime such that  $k_i^0 < k_1 < \cdots < k_m < k_{i+1}^0$ . Consequently, overfitting is an unavoidable and significant issue when discussing asymptotic properties.

We first consider the case of overfitting a point  $\tilde{k}_j$  in regime  $[k_{j-1}^0 + [\tau T], k_j^0 - [\tau T]]$  when  $B_j$  is a singular matrix. Define  $\Sigma(k, s)$  as the covariance matrix of  $Hg_t$ , and let  $\hat{\Sigma}(k, s)$  denote the estimated sample covariance matrix within the regime from k to s, where k < s. Specifically,

$$\hat{\Sigma}(k_{j-1}^{0},k_{j}^{0}) = \hat{G}_{j}'\hat{G}_{j}/(k_{j}^{0}-k_{j-1}^{0}), \ \hat{\Sigma}(k_{j-1}^{0},\tilde{k}_{j}) = \hat{G}_{j,1}'\hat{G}_{j,1}/(\tilde{k}_{j}-k_{j-1}^{0}), \ \hat{\Sigma}(\tilde{k}_{j},k_{j}^{0}) = \hat{G}_{j,2}'\hat{G}_{j,2}/(k_{j}^{0}-\tilde{k}_{j}),$$
where  $\hat{G}_{j} = [\hat{g}_{k_{j-1}^{0}+1},...,\hat{g}_{k_{j}^{0}}]', \ \hat{G}_{j,1} = [\hat{g}_{k_{j-1}^{0}+1},...,\hat{g}_{k_{j}}]', \ \hat{G}_{j,2} = [\hat{g}_{\tilde{k}_{j}+1},...,\hat{g}_{k_{j}^{0}}]'.$  The following theorem establishes the convergence rate of the difference between the eigenvalues of  $\hat{\Sigma}(k_{j-1}^{0},k_{j}^{0})$  and the eigenvalues of  $\hat{\Sigma}(k_{j-1}^{0},\tilde{k}_{j})$ , when  $B_{j}$  is singular. This result is crucial for

the subsequent theoretical analysis.

**Theorem 1** Under Assumptions 1-8 and  $\frac{N}{T} \to \kappa$ , as  $N, T \to \infty$  for  $0 < \kappa < \infty$ , when  $\operatorname{rank}(B_j) = r_j < r$ ,

$$(i) \quad \left| \rho_{\ell} \left( \hat{\Sigma}(k_{j-1}^{0}, k_{j}^{0}) \right) - \rho_{\ell} \left( \hat{\Sigma}(k_{j-1}^{0}, \tilde{k}_{j}) \right) \right| = O_{p} \left( \frac{\log \log N(\tilde{k}_{j} - k_{j-1}^{0})}{N\sqrt{\tilde{k}_{j} - k_{j-1}^{0}}} \right),$$
  
$$(ii) \quad \left| \rho_{\ell} \left( \hat{\Sigma}(k_{j-1}^{0}, k_{j}^{0}) \right) - \rho_{\ell} \left( \hat{\Sigma}(\tilde{k}_{j}, k_{j}^{0}) \right) \right| = O_{p} \left( \frac{\log \log N(k_{j}^{0} - \tilde{k}_{j})}{N\sqrt{k_{j}^{0} - \tilde{k}_{j}}} \right)$$

for  $\ell = r_j + 1, \cdots, r$  uniformly over  $\tilde{k}_j \in [k_{j-1}^0 + r + 1, k_j^0 - r - 1].$ 

In theorem 1, the rate of convergence is determined by N and the sample size of the subsample,  $\tilde{k}_j - k_{j-1}^0$  (or  $k_j^0 - \tilde{k}_j$ ). In the proof of theorem 1, we cannot use the general Central Limit Theorem because the small eigenvalues of the population covariance matrix in

the regime  $[k_{j-1}^0, k_j^0]$  are zero, while the estimated small eigenvalues of the covariance matrix in this regime are  $\delta_{NT}^{-2}$ . This necessitates a more detailed analysis of the asymptotic properties of the singular eigenvalues of the factor covariance matrix. The convergence rate established in Theorem 1 is crucial, as it underpins the consistency of the information criterion used to select the number of breakpoints and ensures the consistency of  $\hat{k}_j$ .

**Theorem 2** Under Assumptions 1–11 and  $\frac{N}{T} \to \kappa$ , as  $N, T \to \infty$  for  $0 < \kappa < \infty$ , (*i*) when  $r_{j,j+1} > \min\{r_j, r_{j+1}\}, \hat{k}_j - k_j = o_p(1);$ (*ii*) when  $r_{j,j+1} = \min\{r_j, r_{j+1}\}, \hat{k}_j - k_j = O_p(1)$ for  $j = 1, \dots, m_0$ , where  $r_{j,j+1} = rank([B_j, B_{j+1}]).$ 

Theorem 2 (i) demonstrates that the estimated change points converge to the true breakpoints w.p.a.1 when  $r_{j,j+1} > \min\{r_j, r_{j+1}\}$  (Type 1). This result is more robust than that of Baltagi et al. (2021), who establish that the distance between estimated and true breakpoints is merely  $O_p(1)$  for Type 1 changes. It is important to note that in Type 1 scenarios, where  $r_{j,j+1} = \max\{r_j, r_{j+1}\}$ , the model accounts for the emergence and disappearance of factors. Our QML estimator proves to be consistent when the number of factors changes acorss regimes, whereas Ma and Su (2018) exclude this case by assumption. Furthermore, Theorem 2 (ii) suggests that the discrepancy between the QML estimators and the true change points is stochastically bounded for Type 2 changes.

To validate these theoretical findings, we conducted simulations involving factor loadings with rotational changes (see DGP 1.D and DGP 1.E in Section 5). The simulation results reveal that our QML estimators exhibit significantly lower MAEs and RMSEs compared to those reported by Baltagi et al. (2021) and Ma and Su (2018).

**Remark 4** Due to the presence of multiple breakpoints, one cannot simply determine the type of breakpoint by counting the number of factors within a regime. For example, in (4), when  $B_j$  is a singular matrix and R is a nonsingular matrix, the number of factors in the regime  $[k_{j-1}, k_{j+1}]$  is rank $(B_j) < r$ . In this case, the difference between the estimated and true breakpoints is bounded, as described by Theorem 2 (ii) for Type 2 breakpoints. However, when  $r > r_{j,j+1} > \min\{r_j, r_{j+1}\}$ , the breakpoint corresponds to Type 1, where the estimated breakpoint is consistent with the true one. In both cases, although  $r > r_{j,j+1}$ , the results are fundamentally different. Similarly, when the number of factors in the regimes  $[k_{j-1}, k_j]$ and  $[k_j, k_{j+1}]$  is the same, we cannot directly determine the nature of the change across the breakpoint. If there exists a nonsingular matrix R such that  $B_{j+1} = B_j R$ , then the breakpoint represents a rotational change (regardless of whether  $B_j$  is full rank or singular), and the estimated breakpoint follows the result of Theorem 2 (ii). If no such matrix R exists, it indicates that the factor loadings before and after the breakpoint are not fully linearly related, and the estimated breakpoint follows the result of Theorem 2 (i). However, in the singlebreak case, the condition  $r > r_{1,2}$  does not arise because such scenarios are excluded during the initial estimation of the number of factors.

**Remark 5** We discuss how to analyze the nature of changes around the breakpoints and assess the reliability of the estimated breakpoint positions. For clarity, let us denote the estimated breakpoints as  $\hat{k}_{j-1}$ ,  $\hat{k}_j$ , and  $\hat{k}_{j+1}$ , and aim to determine the type of change around the breakpoint  $\hat{k}_j$ . First, using our QML estimators and following Bai and Ng's (2002) information criterion, one can estimate the number of factors  $r_j$ ,  $r_{j+1}$ ,  $r_{j,j+1}$  for regimes  $[\hat{k}_{j-1}, \hat{k}_j]$ ,  $[\hat{k}_j, \hat{k}_{j+1}]$ , and  $[\hat{k}_{j-1}, \hat{k}_{j+1}]$ , and the full-sample pseudo factors r.

(A.1) If  $r = \min\{r_j, r_{j+1}\}$  (i.e.,  $r = r_j = r_{j+1} = r_{j,j+1}$ ), then the change across the breakpoint is a full-rank rotational change.

(A.2) If  $r > \min\{r_j, r_{j+1}\}$  and  $r_{j,j+1} = r_j = r_{j+1}$ , then the change is rotational.

(B.1) If  $r > \min\{r_j, r_{j+1}\}$  and  $r_{j,j+1} = r_j + r_{j+1}$ , the factor loading before and after the breakpoint are linearly independent-a scenario commonly assumed in existing literature.

(B.2) If  $r > \min\{r_j, r_{j+1}\}$  and  $r_{j,j+1} = r_j > r_{j+1}$ , the factor loading after the breakpoint are a linear subset of those before the breakpoint, i.e., some factors disappear after the breakpoint.

(B.3) If  $r > \min\{r_j, r_{j+1}\}$  and  $r_{j,j+1} = r_{j+1} < r_j$ , the factor loading before the breakpoint are a linear subset of those after the breakpoint, i.e., new factors emerge after the breakpoint.

For cases (A.1) and (A.2), the difference between the estimated and true breakpoint positions is bounded, as shown by Theorem 2 (ii). In contrast, for cases (B.1), (B.2), and (B.3), the estimated breakpoint positions are consistent with the true breakpoints, as indicated by Theorem 2 (i). Thus, one can assess the type of break around the breakpoint and evaluate the reliability of the estimated breakpoint based on the values of r,  $r_{j,j+1}$ ,  $r_j$  and  $r_{j+1}$ .

#### Determining the number of breaks

We now discuss the choice of the number of breaks m, which is an important issue when the objective function is used in practice. The objective function is monotonically decreasing with respect to m, as proven in Lemma ??. Thus, we propose selecting m to minimize the following information criterion:

$$IC(m) = \sum_{\ell=1}^{m+1} (k_{\ell} - k_{\ell-1}) \log |\hat{\Sigma}(k_{\ell-1}, k_{\ell})| + m(1 + |\hat{\rho}|) r^2 \log(\min(N, T)),$$
(6)

where  $\hat{\rho}$  is the radius of the estimated AR(1) coefficient matrix, obtained by fitting a VAR(1) model to  $\hat{g}_t$ . Let  $m_{max}$  be a bounded integer such that  $m_0 < m_{max}$  and  $\hat{m} = \arg \min_{1 \le m \le m_{max}} IC(m)$ . We add the following assumption.

Assumption 12 The  $r \times r$  matrix  $\frac{\Sigma_{\Lambda}^{-1/2} \Lambda' e'_{k_j}}{N} \cdot \frac{e_{k_j} \Lambda \Sigma_{\Lambda}^{-1/2}}{N} \Sigma_{\Lambda}^{-1/2} \Sigma_F^{-1} \Sigma_{\Lambda}^{-1/2}$  has r different eigenvalues for  $j = 1, \cdots, m_0$ .

Assumption 12 ensures that the smallest eigenvalue of the estimated factor covariance matrix is distinct, with a unique corresponding eigenvector. With this result, we can further conclude that the eigenvalues of the product of the estimated factor covariance matrix and the inverse of the factor covariance matrix from the overfitted subsample are close to an upper triangular matrix with diagonal elements equal to one. For details, see the proof of lemma ??.

**Theorem 3** Under Assumptions 1–10 and 12, and  $\frac{N}{T} \to \kappa$ , as  $N, T \to \infty$  for  $0 < \kappa < \infty$ ,

$$\lim_{N,T\to\infty} P(\hat{m}=m_0)=1.$$

**Remark 6** The penalty function  $(1 + |\hat{\rho}|)r^2 \log(\min(N, T))$  in (6) is designed to ensure the consistent estimation of the number of breaks, applying the same penalty to both types of breaks. This is because overestimating the number of breaks decreases the loss function by a magnitude of  $O_p(1)$ , whereas underestimating the number of breaks increases the loss function by a magnitude of  $O_p(T)$ . The  $|\hat{\rho}|$  term is introduced to prevent the penalty term from being too small due to correlations among the factors, which could otherwise lead to an overestimation of the number of breakpoints.

### 5 Simulation

In this section, we consider DGPs corresponding to different types to evaluate the finite sample performance of the QML estimator. We compare the QML estimator with two other estimators. As shown below,  $\hat{k}_{BKW}$  is the estimator proposed by Baltagi, Kao, and Wang (2021, BKW hereafter);  $\hat{k}_{MS}$  is the estimator proposed by Ma and Su (2018, MS hereafter); and  $\hat{k}_{QML}$  is the QML estimator. The DGP roughly follows BKW, which can be used to examine various elements that may affect the finite sample performance of the estimators, and we use this DGP for model (3). We calculate the root mean square error (RMSE) and mean absolute error (MAE) of these change point estimators  $\hat{k}_{BKW}$ ,  $\hat{k}_{MS}$ , and  $\hat{k}_{QML}$ , and each experiment is repeated 1000 times, where  $\text{RMSE}_i = \sqrt{\frac{1}{1000} \sum_{s=1}^{1000} (\hat{k}_{i,s} - k_i^0)^2}$  and  $\text{MAE}_i = \frac{1}{1000} \sum_{s=1}^{1000} |\hat{k}_{i,s} - k_i^0|$  for  $i = 1, \dots, m_0$ . As the computation of  $\hat{k}_{MS}$  requires the number of original factors and that of  $\hat{k}_{BKW}$  and  $\hat{k}_{QML}$  requires the number of pseudo-factors, we set  $\hat{r} = r_0$  for  $\hat{k}_{MS}$  and  $\hat{r} = r$  for  $\hat{k}_{QML}$  and  $\hat{k}_{BKW}$ , where  $r_0$  is the number of original factors and that of pseudo-factors. Note that the MS method sometimes detects more or fewer than  $m_0$  breaks. For the sake of comparability, we only select the estimated results when exactly  $m_0$  breaks are detected by MS.

Each factor is generated by the following AR(1) process:

$$f_{tp} = \rho f_{t-1,p} + u_{t,p}, \quad for \quad t = 2, \cdots, T; \quad p = 1, \cdots, r_0$$

where  $u_t = (u_{t,1}, \dots, u_{t,r_0})'$  is i.i.d.  $N(0, I_{r_0})$  for  $t = 2, \dots, T$  and  $f_1 = (f_{1,1}, \dots, f_{1,r_0})'$  is i.i.d.  $N(0, \frac{1}{1-\rho^2}I_{r_0})$ . The scalar  $\rho$  captures the serial correlation of factors, and the idiosyncratic errors are generated by

$$e_{i,t} = \alpha e_{i,t-1} + v_{i,t}, \quad for \quad i = 1, \cdots, N; \quad t = 2, \cdots, T,$$

where  $v_t = (v_{1,t}, \dots, v_{N,t})'$  is i.i.d.  $N(0, \Omega)$  for  $t = 2, \dots, T$  and  $e_1 = (e_{1,1}, \dots, e_{N,1})'$  is  $N(0, \frac{1}{(1-\alpha^2)}\Omega)$ . The scalar  $\alpha$  captures the serial correlation of the idiosyncratic errors, and  $\Omega$  is generated as  $\Omega_{ij} = \beta^{|i-j|}$  so that  $\beta$  captures the degree of cross-sectional dependence of the idiosyncratic errors. In addition,  $u_t$  and  $v_t$  are mutually independent for all values of t. We consider the following DGPs for factor loadings and investigate the performance of the QML estimator for different types of breaks discussed in Section 2.

**DGP 1.** We first consider the case in which  $m_0 = 2$  and  $r_0 = 3$ . We set  $k_1^0 = [0.3T]$  and  $k_2^0 = [0.7T]$ , where [·] denotes the rounding operation. This means that there are two breakpoints and three regimes.

**DGP 1.A** We set  $\lambda_{i,0}$  to be i.i.d.  $N(0, \frac{1}{r_0}I_{r_0})$  across i, and define  $\Lambda_0 = (\lambda_{1,0}, \ldots, \lambda_{N,0})'$ . In the first regime,  $\Lambda_1 = (\lambda_{1,1}, \cdots, \lambda_{N,1})' = \Lambda_0 B_1$  with  $B_1 = [1, 0, 0; 0, 1, 0; 0, 0, 0]$ . In the second regime,  $\Lambda_2 = (\lambda_{1,2}, \cdots, \lambda_{N,2})' = \Lambda_0 B_2$  with  $B_2 = [1, 0, 0; 0, 0, 0; 0, 0, 1]$ . In the third regime,  $\Lambda_3 = (\lambda_{1,3}, \cdots, \lambda_{N,3})' = \Lambda_0 B_3$  with  $B_3 = [0, 0, 0; 0, 1, 0; 0, 0, 1]$ . For the full sample, the number of pseudo-factors is r = 3. In each of the three regimes, the number of factors is 2, as rank $(B_1) = \operatorname{rank}(B_2) = \operatorname{rank}(B_3) = 2$ . This setting corresponds to a Type 1 break in Section 2 with  $r_{j,j+1} > \max\{r_j, r_{j+1}\}$ . In this case, there are both independent and correlated factor loadings within adjacent regimes. Table 1 lists the RMSEs and MAEs of these change point estimators  $\hat{k}_{BKW}$ ,  $\hat{k}_{MS}$ , and  $\hat{k}_{QML}$  with different values of  $(\rho, \alpha, \beta)$ .

In all cases,  $\hat{k}_{QML}$  has much smaller RMSEs and MAEs than  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$ , and the RMSEs and MAEs of  $\hat{k}_{QML}$  tend to decrease as N and T increase. This confirms the consistency of  $\hat{k}_{QML}$  established in Theorem 2 (i). In addition, the RMSEs and MAEs of  $\hat{k}_{BKW}$  do not converge to zero as N and T increase, indicating that the estimation error of  $\hat{k}_{BKW}$  remains stochastically bounded. Furthermore, a larger AR(1) coefficient  $\rho$  tends to worsen the performance of  $\hat{k}_{BKW}$ , while it does not have effect on our QML estimator. As the sample size increases, although the performance of  $\hat{k}_{MS}$  gradually improves, it still falls short of QML in terms of RMSEs and MAEs.

Table 1: Simulated root mean squared errors (RMSEs) and mean absolute errors (MAEs) of  $\hat{k}_{QML}$ ,  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$ , under DGP 1.A. The last column, "Num," represents the number of times the MS method correctly estimated two breakpoints in 1000 iterations.

| N, T    | $(\hat{k}_1,\hat{k}_2)$ | $_{2})_{QML}$  | $(\hat{k}_1, \hat{k}_2)_{BKW}$ $(\hat{k}_1, \hat{k}_2)_{MS}$ |                  |                  |                  |     |
|---------|-------------------------|----------------|--|------------------|------------------|------------------|-----|
|         | RMSE                    | MAE            | RMSE   | MAE              | RMSE             | MAE              | Num |
|         |                         |                | $\rho = 0$   | $\alpha = 0$     | $\beta = 0$      |                  |     |
| 100,100 | (0.585, 0.587)          | (0.238, 0.220) | (10.278, 10.950)   | (4.934, 5.411)   | (13.527, 16.188) | (9.112, 11.066)  | 197 |
| 100,300 | (0.492, 0.542)          | (0.194, 0.216) | (3.754, 2.906)   | (1.323, 1.283)   | (38.707, 40.302) | (23.927, 27.641) | 220 |
| 300,300 | (0.366, 0.355)          | (0.114, 0.108) | (3.459, 3.672)   | (1.275, 1.251)   | (32.591, 32.323) | (19.131, 18.664) | 444 |
| 300,600 | (0.354, 0.372)          | (0.107, 0.118) | (1.899, 1.991)   | (0.886, 0.880)   | (30.847, 33.765) | (9.957, 9.870)   | 563 |
| 600,600 | (0.290, 0.293)          | (0.080, 0.076) | (1.967, 1.943)   | (0.864, 0.864)   | (8.324, 17.133)  | (1.518, 2.590)   | 566 |
|         |                         |                | $\rho = 0.7$   | $\alpha = 0$     | $\beta = 0$      |                  |     |
| 100,100 | (0.575, 0.495)          | (0.199, 0.173) | (20.072, 19.915)   | (14.724, 14.257) | (11.472, 12.697) | (7.487, 7.809)   | 230 |
| 100,300 | (0.504, 0.427)          | (0.180, 0.136) | (30.374, 30.487)   | (13.404, 12.818) | (31.692, 31.232) | (17.584, 17.151) | 449 |
| 300,300 | (0.316, 0.363)          | (0.088, 0.106) | (27.303, 27.966)   | (11.891, 11.699) | (21.725, 23.120) | (9.920, 8.905)   | 513 |
| 300,600 | (0.315, 0.305)          | (0.081, 0.075) | (15.284, 15.701)   | (4.036, 3.498)   | (9.122, 14.798)  | (1.736, 2.338)   | 462 |
| 600,600 | (0.221, 0.261)          | (0.041, 0.060) | (22.984, 16.870)   | (5.098, 3.842)   | (1.162, 2.937)   | (0.662, 0.671)   | 459 |
|         |                         |                | $\rho = 0$   | $\alpha = 0.3$   | $\beta = 0$      |                  |     |
| 100,100 | (0.636, 0.578)          | (0.268, 0.232) | (11.187, 10.698)   | (5.679, 5.243)   | (14.503, 16.041) | (10.362, 10.710) | 221 |
| 100,300 | (0.551, 0.611)          | (0.210, 0.247) | (4.328, 4.518)   | (1.482, 1.537)   | (34.096, 33.528) | (22.185, 21.263) | 259 |
| 300,300 | (0.417, 0.356)          | (0.130, 0.111) | (2.625, 2.818)   | (1.122, 1.237)   | (33.489, 27.816) | (19.837, 15.085) | 436 |
| 300,600 | (0.370, 0.369)          | (0.117, 0.110) | (2.049, 1.685)   | (0.922, 0.757)   | (31.595, 28.276) | (9.600, 8.324)   | 547 |
| 600,600 | (0.329, 0.313)          | (0.094, 0.088) | (2.127, 1.826)   | (0.937, 0.825)   | (11.666, 6.868)  | (2.513, 1.230)   | 534 |
|         |                         |                | $\rho = 0$   | $\alpha = 0$     | $\beta = 0.3$    |                  |     |
| 100,100 | (0.576, 0.680)          | (0.240, 0.282) | (10.678, 11.051)   | (5.083, 5.397)   | (14.617, 16.179) | (10.349, 10.967) | 212 |
| 100,300 | (0.624, 0.551)          | (0.249, 0.225) | (3.181, 2.893)   | (1.410, 1.323)   | (36.529, 31.525) | (23.806, 19.872) | 242 |
| 300,300 | (0.435, 0.329)          | (0.143, 0.096) | (2.696, 2.752)   | (1.227, 1.146)   | (28.129, 31.622) | (16.158, 16.622) | 437 |
| 300,600 | (0.362, 0.363)          | (0.113, 0.116) | (2.100, 1.877)   | (0.939, 0.780)   | (31.471, 37.708) | (10.541, 10.886) | 545 |
| 600,600 | (0.311, 0.300)          | (0.087, 0.082) | (2.034, 1.865)   | (0.898, 0.833)   | (13.322, 6.358)  | (2.953, 1.107)   | 552 |
|         |                         |                | $\rho = 0.7$   | $\alpha = 0.3$   | $\beta = 0.3$    |                  |     |
| 100,100 | (0.531, 0.577)          | (0.192, 0.179) | (21.127, 19.862)   | (15.306, 14.535) | (12.497, 13.936) | (8.482, 8.872)   | 243 |
| 100,300 | (0.504, 0.427)          | (0.180, 0.136) | (30.374, 30.487)   | (13.404, 12.818) | (31.692, 31.232) | (17.584, 17.151) | 449 |
| 300,300 | (0.344, 0.336)          | (0.082, 0.089) | (30.739, 26.776)   | (12.881, 11.394) | (23.455, 21.124) | (10.231, 7.694)  | 545 |
| 300,600 | (0.295, 0.321)          | (0.077, 0.081) | (16.831, 17.168)   | (4.216, 3.952)   | (5.083, 6.411)   | (1.088, 1.197)   | 456 |
| 600,600 | (0.221, 0.307)          | (0.049, 0.074) | (15.492, 18.497)   | (3.935, 4.216)   | (3.111, 3.017)   | (0.836, 0.587)   | 421 |

#### DGP 1.B

In this setup, we adopt the DGP from MS. We set  $\Lambda_1 = (\lambda_{1,1}, \ldots, \lambda_{N,1})'$ , where  $\lambda_{i,1}$  are i.i.d.  $N((0.5b, 0.5b, 0.5b), \frac{1}{r_0}I_{r_0})$  across *i* in the first regime. In the second regime,  $\Lambda_2 =$  $(\lambda_{1,2}, \cdots, \lambda_{N,2})'$  and  $\lambda_{i,2}$  are i.i.d.  $N((b,b,b), \frac{1}{r_0}I_{r_0})$  across *i*. In the third regime,  $\Lambda_3 =$  $(\lambda_{1,3},\cdots,\lambda_{N,3})'$  and  $\lambda_{i,3}$  are i.i.d.  $N((1.5b, 1.5b, 1.5b), \frac{1}{r_0}I_{r_0})$  across *i*. The number of pseudofactors for the full samples in this case is r = 9, and the numbers of factors in each regime is 3. This setting corresponds to the case in Type 1 with  $r_{j,j+1} = r_j + r_{j+1}$ . In this case, the factor loadings within adjacent regimes are independent. Different values of b can be used to control the scale of change between regimes. Tables 2 and 3 present the RMSEs and MAEs of these change point estimators  $\hat{k}_{BKW}$ ,  $\hat{k}_{MS}$ , and  $\hat{k}_{QML}$  with different values of  $(\rho, \alpha, \beta)$ , corresponding to b = 1 and b = 0, respectively. From Tables 2 and 3, we observe that the RMSEs and MAEs of the  $k_{QML}$  estimator are close to zero, even with a small sample size, indicating that the QML method can nearly accurately identify the breakpoints. Furthermore, as the sample size increases, the RMSEs and MAEs of the QML estimator converge to zero, further validating the consistency of Theorem 2 (i).  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$  estimators also perform well, but they are still not as accurate as the QML estimation method.

Table 2: Simulated root mean squared errors (RMSEs) and mean absolute errors (MAEs) of  $\hat{k}_{QML}$ ,  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$ , under DGP 1.B with b = 1. The last column, "Num," represents the number of times the MS method correctly estimated two breakpoints in 1000 iterations.

| N, T    | $(\hat{k}_1, \hat{k}_2)_{QML}$ $(\hat{k}_1, \hat{k}_2)_{BKW}$ |                | $(\hat{k}$     | $(\hat{k}_1,\hat{k}_2)_{MS}$ |                |                |     |
|---------|---|----------------|----------------|------------------------------|----------------|----------------|-----|
|         | RMSE  | MAE            | RMSE           | MAE                          | RMSE           | MAE            | Num |
|         |   |                | $\rho = 0$     | $\alpha = 0$                 | $\beta = 0$    |                |     |
| 100,100 | (0.148, 0.134)  | (0.022, 0.018) | (1.166, 1.261) | (0.356, 0.335)               | (1.396, 6.403) | (0.550, 1.500) | 40  |
| 100,300 | (0.110, 0.148)  | (0.012, 0.018) | (0.586, 0.688) | (0.205, 0.233)               | (0.427, 0.334) | (0.097, 0.063) | 896 |
| 300,300 | (0.055, 0.055)  | (0.003, 0.003) | (0.696, 0.638) | (0.243, 0.215)               | (0.554, 0.502) | (0.156, 0.126) | 799 |
| 300,600 | (0.055, 0.045)  | (0.003, 0.002) | (0.625, 0.612) | (0.215, 0.208)               | (0.685, 0.653) | (0.236, 0.213) | 798 |
| 600,600 | (0.000, 0.032)  | (0.000, 0.001) | (0.611, 0.538) | (0.213, 0.185)               | (0.825, 0.687) | (0.340, 0.236) | 711 |
|         |   |                | $\rho = 0.7$   | $\alpha = 0$                 | $\beta = 0$    |                |     |
| 100,100 | (0.078, 0.078)  | (0.006, 0.006) | (2.675, 3.631) | (0.702, 0.926)               | (2.894, 6.358) | (0.837, 1.767) | 43  |
| 100,300 | (0.063, 0.071)  | (0.004, 0.005) | (0.987, 1.883) | (0.347, 0.455)               | (0.732, 0.590) | (0.270, 0.176) | 790 |
| 300,300 | (0.000, 0.032)  | (0.000, 0.001) | (1.569, 1.072) | (0.329, 0.341)               | (0.902, 0.660) | (0.407, 0.218) | 688 |
| 300,600 | (0.045, 0.045)  | (0.002, 0.002) | (0.808, 1.002) | (0.258, 0.303)               | (0.967, 0.769) | (0.466, 0.297) | 644 |
| 600,600 | (0.000, 0.032)  | (0.000, 0.001) | (1.084, 1.079) | (0.368, 0.337)               | (1.061, 0.867) | (0.563, 0.377) | 597 |
|         |   |                | $\rho = 0$     | $\alpha = 0.3$               | $\beta = 0$    |                |     |
| 100,100 | (0.170, 0.164)  | (0.025, 0.025) | (0.921, 1.165) | (0.300, 0.321)               | (0.697, 0.775) | (0.257, 0.314) | 35  |
| 100,300 | (0.138, 0.114)  | (0.019, 0.013) | (0.636, 0.748) | (0.231, 0.257)               | (0.431, 0.324) | (0.102, 0.059) | 906 |
| 300,300 | (0.055, 0.000)  | (0.003, 0.000) | (0.665, 0.683) | (0.228, 0.248)               | (0.600, 0.516) | (0.182, 0.134) | 801 |
| 300,600 | (0.032, 0.045)  | (0.001, 0.002) | (0.636, 0.528) | (0.217, 0.177)               | (0.687, 0.661) | (0.237, 0.217) | 782 |
| 600,600 | (0.032, 0.045)  | (0.001, 0.002) | (0.528, 0.595) | (0.187, 0.188)               | (0.863, 0.742) | (0.373, 0.275) | 698 |
|         |   |                | $\rho = 0$     | $\alpha = 0$                 | $\beta = 0.3$  |                |     |
| 100,100 | (0.145, 0.148)  | (0.019, 0.022) | (1.318, 1.157) | (0.338, 0.339)               | (3.919, 6.342) | (1.167, 1.262) | 42  |
| 100,300 | (0.100, 0.118)  | (0.010, 0.014) | (0.660, 0.842) | (0.212, 0.286)               | (0.381, 0.404) | (0.077, 0.088) | 882 |
| 300,300 | (0.078, 0.055)  | (0.006, 0.003) | (0.593, 0.642) | (0.211, 0.210)               | (0.547, 0.522) | (0.152, 0.137) | 815 |
| 300,600 | (0.105, 0.032)  | (0.011, 0.001) | (0.623, 0.573) | (0.202, 0.196)               | (0.772, 0.648) | (0.302, 0.210) | 782 |
| 600,600 | (0.032, 0.000)  | (0.001, 0.000) | (0.569, 0.536) | (0.198, 0.191)               | (0.818, 0.678) | (0.334, 0.230) | 730 |
|         |   |                | $\rho = 0.7$   | $\alpha = 0.3$               | $\beta = 0.3$  |                |     |
| 100,100 | (0.118, 0.095)  | (0.012, 0.009) | (2.471, 3.445) | (0.650, 0.873)               | (2.121, 5.266) | (0.750, 1.075) | 40  |
| 100,300 | (0.055, 0.055)  | (0.003, 0.003) | (2.317, 1.338) | (0.418, 0.409)               | (0.744, 0.532) | (0.278, 0.143) | 785 |
| 300,300 | (0.000, 0.055)  | (0.000, 0.003) | (1.167, 1.200) | (0.364, 0.372)               | (0.969, 0.693) | (0.469, 0.241) | 686 |
| 300,600 | (0.045, 0.032)  | (0.002, 0.001) | (1.046, 1.051) | (0.358, 0.335)               | (0.985, 0.823) | (0.486, 0.339) | 632 |
| 600,600 | (0.032, 0.032)  | (0.001, 0.001) | (0.818, 1.030) | (0.271, 0.319)               | (1.044, 0.870) | (0.543, 0.379) | 567 |

Table 3: Simulated root mean squared errors (RMSEs) and mean absolute errors (MAEs) of  $\hat{k}_{QML}$ ,  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$ , under DGP 1.B with b = 0. The last column, "Num," represents the number of times the MS method correctly estimated two breakpoints in 1000 iterations.

| N, T    | $(\hat{k}_1, \hat{k}_2)_{QML}$ $(\hat{k}_1, \hat{k}_2)_{BKW}$ |                | $(\hat{k}_1,\hat{k}_2)_{MS}$ |                |                |                |     |
|---------|---|----------------|------------------------------|----------------|----------------|----------------|-----|
|         | RMSE  | MAE            | RMSE                         | MAE            | RMSE           | MAE            | Num |
|         |   |                | $\rho = 0$                   | $\alpha = 0$   | $\beta = 0$    |                |     |
| 100,100 | (0.182, 0.195)  | (0.033, 0.036) | (1.017, 0.903)               | (0.285, 0.299) | (1.265, 0.894) | (0.800, 0.400) | 5   |
| 100,300 | (0.173, 0.158)  | (0.022, 0.023) | (0.706, 0.621)               | (0.245, 0.225) | (2.652, 2.956) | (0.417, 0.411) | 696 |
| 300,300 | (0.055, 0.078)  | (0.003, 0.006) | (0.691, 0.575)               | (0.228, 0.196) | (0.951, 0.787) | (0.453, 0.312) | 404 |
| 300,600 | (0.063, 0.045)  | (0.004, 0.002) | (0.519, 0.459)               | (0.175, 0.153) | (1.177, 0.923) | (0.693, 0.425) | 407 |
| 600,600 | (0.045, 0.045)  | (0.002, 0.002) | (0.533, 0.544)               | (0.178, 0.170) | (1.270, 0.939) | (0.806, 0.441) | 263 |
|         |   |                | $\rho = 0.7$                 | $\alpha = 0$   | $\beta = 0$    |                |     |
| 100,100 | (0.089, 0.114)  | (0.008, 0.011) | (2.963, 2.075)               | (0.765, 0.492) | (3.691, 1.581) | (2.375, 1.250) | 8   |
| 100,300 | (0.095, 0.055)  | (0.009, 0.003) | (1.169, 1.752)               | (0.338, 0.397) | (0.931, 0.805) | (0.432, 0.326) | 565 |
| 300,300 | (0.032, 0.000)  | (0.001, 0.000) | (1.238, 1.157)               | (0.364, 0.312) | (1.060, 0.922) | (0.561, 0.425) | 367 |
| 300,600 | (0.032, 0.045)  | (0.001, 0.002) | (0.865, 0.909)               | (0.265, 0.260) | (1.350, 0.934) | (0.912, 0.438) | 226 |
| 600,600 | (0.000, 0.045)  | (0.000, 0.002) | (0.932, 0.922)               | (0.275, 0.294) | (1.257, 0.971) | (0.790, 0.471) | 157 |
|         |   |                | $\rho = 0$                   | $\alpha = 0.3$ | $\beta = 0$    |                |     |
| 100,100 | (0.251, 0.205)  | (0.057, 0.040) | (1.041, 1.194)               | (0.303, 0.322) | (4.000, 5.344) | (1.333, 3.000) | 9   |
| 100,300 | (0.148, 0.155)  | (0.022, 0.022) | (0.676, 0.659)               | (0.241, 0.226) | (3.437, 1.814) | (0.482, 0.389) | 691 |
| 300,300 | (0.063, 0.071)  | (0.004, 0.005) | (0.562, 0.580)               | (0.190, 0.206) | (1.082, 0.781) | (0.586, 0.302) | 377 |
| 300,600 | (0.055, 0.063)  | (0.003, 0.004) | (0.509, 0.465)               | (0.169, 0.146) | (1.196, 0.958) | (0.715, 0.463) | 400 |
| 600,600 | (0.055, 0.045)  | (0.003, 0.002) | (0.501, 0.506)               | (0.175, 0.150) | (1.294, 0.909) | (0.838, 0.408) | 277 |
|         |   |                | $\rho = 0$                   | $\alpha = 0$   | $\beta = 0.3$  |                |     |
| 100,100 | (0.179, 0.192)  | (0.032, 0.033) | (1.102, 1.111)               | (0.287, 0.286) | (1.069, 0.756) | (0.571, 0.286) | 7   |
| 100,300 | (0.145, 0.176)  | (0.021, 0.031) | (0.622, 0.656)               | (0.197, 0.218) | (5.322, 3.584) | (0.561, 0.521) | 684 |
| 300,300 | (0.055, 0.045)  | (0.003, 0.002) | (0.546, 0.610)               | (0.182, 0.194) | (1.024, 0.836) | (0.526, 0.352) | 386 |
| 300,600 | (0.055, 0.045)  | (0.003, 0.002) | (0.496, 0.431)               | (0.162, 0.134) | (1.203, 1.023) | (0.724, 0.525) | 406 |
| 600,600 | (0.055, 0.045)  | (0.003, 0.002) | (0.507, 0.503)               | (0.175, 0.149) | (1.255, 0.893) | (0.788, 0.401) | 287 |
|         |   |                | $\rho = 0.7$                 | $\alpha = 0.3$ | $\beta = 0.3$  |                |     |
| 100,100 | (0.100, 0.114)  | (0.008, 0.011) | (3.115, 2.989)               | (0.788, 0.718) | (4.336, 1.265) | (2.800, 0.800) | 5   |
| 100,300 | (0.089, 0.141)  | (0.008, 0.012) | (1.159, 0.967)               | (0.342, 0.298) | (0.933, 0.880) | (0.436, 0.389) | 530 |
| 300,300 | (0.000, 0.055)  | (0.000, 0.003) | (1.099, 2.137)               | (0.366, 0.328) | (1.153, 0.920) | (0.665, 0.424) | 304 |
| 300,600 | (0.032, 0.032)  | (0.001, 0.001) | (0.921, 1.050)               | (0.269, 0.304) | (1.298, 0.943) | (0.843, 0.444) | 216 |
| 600,600 | (0.032, 0.000)  | (0.001, 0.000) | (0.788, 0.980)               | (0.237, 0.255) | (1.429, 0.916) | (1.021, 0.420) | 143 |

**DGP 1.C** We set  $\lambda_{i,0}$  are i.i.d.  $N(0, \frac{1}{r_0}I_{r_0})$  across i and  $\Lambda_0 = (\lambda_{1,0}, \cdots, \lambda_{N,0})'$ . In the first regime,  $\Lambda_1 = (\lambda_{1,1}, \dots, \lambda_{N,1})' = \Lambda_0 B_1$  with  $B_1 = [1, 0, 0; 0, 1, 0; 0, 0, 1]$ . In the second regime,  $\Lambda_2 = (\lambda_{1,2}, \dots, \lambda_{N,2})' = \Lambda_0 B_2$  with  $B_2 = [1, 0, 0; 0, 1; 0, 0, 0]$ . In the third regime,  $\Lambda_3 = (\lambda_{1,3}, \dots, \lambda_{N,3})' = \Lambda_0 B_3$  with  $B_3 = [0, 0, 0; 0, 0, 0; 0, 0, 1]$ . The number of pseudo-factors for the full sample is r = 3, with the numbers of factors in the three regimes being 3, 2, and 1, respectively, since  $\operatorname{rank}(B_1) = 3$ ,  $\operatorname{rank}(B_2) = 2$ , and  $\operatorname{rank}(B_3) = 1$ . Table 4 lists the RMSEs and MAEs of three estimators. In the first regime, all three factors are present. In the second regime, only the first two factors remain, with the third factor disappearing. In the third regime, only the third factor remains, while the first two factors vanish. This setting corresponds to the case in Type 1 of Section 2, where some factors disappear after some break points and  $r_{j,j+1} = \max\{r_j, r_{j+1}\}$ . From Table 4, we observe that when the factors are serially independent ( $\rho = 0$ ), the BKW method works, though there is significant bias in the estimated results. However, when the factors are correlated ( $\rho = 0.7$ ), the BKW method performs worse. The MS method also seems to struggle in this case, as both the RMSEs and MAEs increase with the growing sample size. This is because the MS method assumes a constant number of factors throughout. Nevertheless,  $\hat{k}_{QML}$  still performs the best, and as the sample size increases, the RMSEs and MAEs gradually decrease.

Table 4: Simulated root mean squared errors (RMSEs) and mean absolute errors (MAEs) of  $\hat{k}_{QML}$ ,  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$ , under DGP 1.C. The last column, "Num," represents the number of times the MS method correctly estimated two breakpoints in 1000 iterations.

| N, T    | $(\hat{k}_1, \hat{k}_2)$ | $_{2})_{QML}$  | $(\hat{k}_1,\hat{k}_2$ | $)_{BKW}$        | $(\hat{k}_1,\hat{k}_2)_{MS}$ |                  |     |
|---------|--------------------------|----------------|------------------------|------------------|------------------------------|------------------|-----|
|         | RMSE                     | MAE            | RMSE                   | MAE              | RMSE                         | MAE              | Num |
|         |                          |                | $\rho = 0$             | $\alpha = 0$     | $\beta = 0$                  |                  |     |
| 100,100 | (1.016, 0.336)           | (0.473, 0.097) | (11.441, 9.371)        | (7.917, 3.235)   | (17.860, 5.496)              | (12.216, 2.775)  | 102 |
| 100,300 | (0.915, 0.302)           | (0.436, 0.077) | (12.283, 1.804)        | (6.219, 0.528)   | (33.254, 15.893)             | (24.851, 4.140)  | 443 |
| 300,300 | (0.623, 0.152)           | (0.238, 0.023) | (10.671, 1.417)        | (5.222, 0.425)   | (42.555, 4.143)              | (28.226, 0.878)  | 204 |
| 300,600 | (0.525, 0.176)           | (0.202, 0.029) | (7.939, 0.807)         | (3.896, 0.319)   | (69.553, 5.166)              | (52.161, 1.666)  | 323 |
| 600,600 | (0.428, 0.084)           | (0.147, 0.007) | (8.311, 0.773)         | (4.289, 0.289)   | (80.610, 0.244)              | (65.299, 0.030)  | 67  |
|         |                          |                | $\rho = 0.7$           | $\alpha = 0$     | $\beta = 0$                  |                  |     |
| 100,100 | (0.932, 0.272)           | (0.373, 0.058) | (18.067, 21.688)       | (14.712, 14.044) | (14.375, 5.354)              | (9.513, 2.835)   | 236 |
| 100,300 | (0.925, 0.207)           | (0.355, 0.035) | (33.006, 21.240)       | (20.774, 6.111)  | (31.273, 25.690)             | (23.129, 9.708)  | 489 |
| 300,300 | (0.519, 0.100)           | (0.179, 0.010) | (33.220, 20.436)       | (20.490, 5.413)  | (35.426, 3.043)              | (25.134,  0.769) | 134 |
| 300,600 | (0.503, 0.105)           | (0.171, 0.011) | (33.781, 4.921)        | (17.369, 1.024)  | (71.489, 1.827)              | (53.703, 1.585)  | 219 |
| 600,600 | (0.358, 0.105)           | (0.100, 0.009) | (34.375, 7.976)        | (16.771, 1.160)  | (96.833, 0.316)              | (83.300, 0.100)  | 10  |
|         |                          |                | $\rho = 0$             | $\alpha = 0.3$   | $\beta = 0$                  |                  |     |
| 100,100 | (1.146, 0.358)           | (0.548, 0.104) | (12.478, 8.471)        | (8.224, 3.000)   | (19.026, 5.281)              | (13.401, 2.918)  | 147 |
| 100,300 | (1.019, 0.310)           | (0.449, 0.080) | (10.994, 2.340)        | (5.665, 0.500)   | $(32.237,\!17.604)$          | (25.454, 4.403)  | 414 |
| 300,300 | (0.584, 0.182)           | (0.239, 0.031) | (12.211, 2.303)        | (5.772, 0.503)   | (34.548, 1.604)              | (25.883, 1.120)  | 460 |
| 300,600 | (0.589, 0.182)           | (0.237, 0.031) | (8.609, 0.755)         | (4.184, 0.288)   | (59.920, 1.953)              | (44.780, 1.770)  | 587 |
| 600,600 | (0.449, 0.095)           | (0.152, 0.009) | (8.119, 0.898)         | (3.966, 0.333)   | (61.981, 2.037)              | (47.405, 1.917)  | 482 |
|         |                          |                | $\rho = 0$             | $\alpha = 0$     | $\beta = 0.3$                |                  |     |
| 100,100 | (0.996, 0.300)           | (0.452, 0.080) | (12.469, 7.292)        | (8.288, 2.384)   | (15.848, 4.817)              | (10.581, 2.675)  | 117 |
| 100,300 | (0.998, 0.305)           | (0.449, 0.085) | (9.940, 1.243)         | (5.155, 0.475)   | (32.588, 20.696)             | (25.027, 6.211)  | 402 |
| 300,300 | (0.621, 0.190)           | (0.229, 0.034) | (11.364, 1.101)        | (5.958, 0.438)   | (38.026, 4.607)              | (22.914, 1.177)  | 232 |
| 300,600 | (0.550, 0.170)           | (0.224, 0.027) | (8.689, 0.784)         | (4.202, 0.298)   | (71.898, 2.406)              | (53.062, 1.562)  | 422 |
| 600,600 | (0.458, 0.114)           | (0.160, 0.013) | (8.849, 0.912)         | (4.247, 0.354)   | (70.002, 0.633)              | (55.300, 0.200)  | 40  |
|         |                          |                | $\rho = 0.7$           | $\alpha = 0.3$   | $\beta = 0.3$                |                  |     |
| 100,100 | (1.097, 0.249)           | (0.467, 0.054) | (17.995, 21.305)       | (14.608, 13.565) | (14.184, 3.582)              | (9.871, 1.809)   | 278 |
| 100,300 | (0.982, 0.278)           | (0.412, 0.053) | (35.058, 24.497)       | (22.097, 7.168)  | (32.378, 24.107)             | (25.337, 9.121)  | 514 |
| 300,300 | (0.602, 0.138)           | (0.198, 0.019) | (31.680, 20.713)       | (19.468, 5.404)  | (33.039, 2.012)              | (25.856, 1.414)  | 423 |
| 300,600 | (0.518, 0.127)           | (0.180, 0.016) | (35.115, 8.706)        | (17.147, 1.419)  | (53.368, 1.914)              | (38.673, 1.619)  | 551 |
| 600,600 | (0.486, 0.123)           | (0.150, 0.013) | (31.109, 9.982)        | (15.349, 1.221)  | (59.029, 2.181)              | (45.495, 1.841)  | 428 |

**DGP 1.D** Let  $\Lambda_1 = \Lambda_0$ ,  $\Lambda_2 = 2\Lambda_0$  and  $\Lambda_3 = \Lambda_0$ . In this case, the number of factors in each regime is the same, and the changes between the factor loadings are only in their multiples, meaning they undergo full rank rotational changes. This setting corresponds to a Type 2 break with  $r = r_{j,j+1} = r_j = r_{j+1}$ . In this case, the factor loadings within adjacent regimes are perfectly linearly correlated. Table 6 reports the performance of these estimators. From the table, we can see that the serial correlation between factors greatly affects the BKW method. When  $\rho = 0.7$ , the RMSEs and MAEs of  $\hat{k}_{BKW}$  are much larger than when  $\rho = 0$ . The MS method does not generate accurate estimates in this case, as both RMSEs and MAEs gradually increase with the sample size. Once again, the QML method remains effective, which also verifies our Theorem 2 (ii) that the difference between the QML estimator and the true breakpoints is bounded under rotational change. Although  $\hat{k}_{BKW}$  also reaches the same conclusion for this type of break, QML estimator clearly outperforms their estimator according to the simulation results.

Table 5: Simulated root mean squared errors (RMSEs) and mean absolute errors (MAEs) of  $\hat{k}_{QML}$ ,  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$ , under DGP 1.D. The last column, "Num," represents the number of times the MS method correctly estimated two breakpoints in 1000 iterations.

| N, T    | $(\hat{k}_1,\hat{k}_2)$ | $_{2})_{QML}$  | $(\hat{k}_1,\hat{k}_2$ | $(\hat{k}_1, \hat{k}_2)_{BKW}$ $(\hat{k}_1, \hat{k}_2)_{MS}$ |                  |                     |     |
|---------|-------------------------|----------------|------------------------|--|------------------|---------------------|-----|
|         | RMSE                    | MAE            | RMSE                   | MAE  | RMSE             | MAE                 | Num |
|         |                         |                | $\rho = 0$             | $\alpha = 0$   | $\beta = 0$      |                     |     |
| 100,100 | (3.496, 3.109)          | (1.575, 1.619) | (10.317, 10.069)       | (6.108, 5.930)   | (20.658, 20.860) | (17.500, 16.750)    | 80  |
| 100,300 | (2.054, 2.109)          | (1.159, 1.142) | (6.035, 5.698)         | (2.851, 2.448)   | (47.356, 42.423) | (37.796, 35.259)    | 402 |
| 300,300 | (2.215, 2.111)          | (1.184, 1.173) | (6.115, 5.521)         | (2.805, 2.475)   | (47.817, 41.938) | (38.664, 34.751)    | 425 |
| 300,600 | (1.888, 1.859)          | (1.859, 1.006) | (4.273, 4.993)         | (2.078, 2.368)   | (94.145, 95.593) | (76.831, 77.552)    | 326 |
| 600,600 | (2.165, 1.987)          | (1.063, 1.017) | (5.295, 3.856)         | (2.442, 2.000)   | (95.014, 86.729) | (77.629, 71.579)    | 202 |
|         |                         |                | $\rho = 0.7$           | $\alpha = 0$   | $\beta = 0$      |                     |     |
| 100,100 | (8.479, 8.603)          | (4.521, 4.592) | (15.649, 16.261)       | (11.746, 12.495)   | (18.690, 18.562) | (15.064, 15.018)    | 109 |
| 100,300 | (5.341, 6.392)          | (2.765, 3.037) | (28.486, 29.621)       | (15.658, 16.752)   | (46.643, 45.121) | $(38.320,\!37.091)$ | 416 |
| 300,300 | (6.208, 5.149)          | (3.050, 2.782) | (29.165, 27.955)       | (16.648, 15.486)   | (46.087, 46.810) | (37.729, 38.957)    | 299 |
| 300,600 | (5.058, 5.173)          | (2.553, 2.670) | (22.849, 23.002)       | (8.809, 9.277)   | (91.392, 93.097) | (71.639, 75.528)    | 144 |
| 600,600 | (5.695, 5.415)          | (2.832, 2.716) | (20.543, 19.428)       | (8.459, 7.793)   | (88.560, 92.720) | (68.569, 75.343)    | 102 |
|         |                         |                | $\rho = 0$             | $\alpha = 0.3$   | $\beta = 0$      |                     |     |
| 100,100 | (3.529, 3.143)          | (1.612, 1.657) | (10.447, 10.092)       | (6.234, 5.912)   | (19.241, 17.709) | (16.067, 13.640)    | 75  |
| 100,300 | (2.057, 2.030)          | (1.145, 1.093) | (5.867, 5.986)         | (2.715, 2.528)   | (47.429, 44.239) | (37.707, 36.320)    | 403 |
| 300,300 | (2.225, 2.114)          | (1.192, 1.167) | (6.159, 5.722)         | (2.805, 2.554)   | (47.924, 44.531) | (38.340, 36.859)    | 412 |
| 300,600 | (1.938, 1.895)          | (1.026, 1.027) | (4.256, 4.964)         | (2.126, 2.354)   | (96.138, 96.734) | $(78.785,\!80.658)$ | 330 |
| 600,600 | (1.846, 1.993)          | (0.987, 1.055) | (4.536, 4.712)         | (2.252, 2.201)   | (98.905, 94.103) | (82.469, 76.454)    | 194 |
|         |                         |                | $\rho = 0$             | $\alpha = 0$   | $\beta = 0.3$    |                     |     |
| 100,100 | (3.358, 3.157)          | (1.486, 1.667) | (10.340, 10.413)       | (6.117, 6.219)   | (21.086, 19.073) | (18.085, 15.451)    | 71  |
| 100,300 | (3.365, 3.380)          | (1.560, 1.641) | (10.121, 9.705)        | (6.014, 5.840)   | (20.953, 17.881) | (18.039, 15.115)    | 26  |
| 300,300 | (2.148, 2.093)          | (1.170, 1.163) | (6.112, 5.441)         | (2.796, 2.466)   | (48.737, 42.597) | (39.724, 35.161)    | 416 |
| 300,600 | (1.912, 1.882)          | (1.003, 1.009) | (4.229, 4.924)         | (2.081, 2.317)   | (96.557, 92.987) | (78.411, 76.160)    | 319 |
| 600,600 | (1.924, 2.038)          | (1.012, 1.086) | (4.574, 4.716)         | (2.276, 2.222)   | (97.734, 91.644) | (81.787, 74.427)    | 192 |
|         |                         |                | $\rho = 0.7$           | $\alpha = 0.3$   | $\beta = 0.3$    |                     |     |
| 100,100 | (8.565, 8.408)          | (4.572, 4.490) | (15.741, 16.328)       | (11.771, 12.532)   | (16.173, 16.614) | (13.441, 13.643)    | 84  |
| 100,300 | (5.619, 6.389)          | (2.914, 3.094) | (28.825, 30.149)       | (16.010, 17.198)   | (45.309, 44.115) | $(36.507,\!36.493)$ | 406 |
| 300,300 | (6.326, 5.310)          | (3.101, 2.892) | (29.341, 28.247)       | (16.790, 15.777)   | (41.986, 44.212) | (33.615, 36.755)    | 273 |
| 300,600 | (4.921, 5.171)          | (2.499, 2.685) | (22.760, 22.752)       | (8.788, 9.281)   | (93.980, 84.635) | $(74.508,\!67.463)$ | 134 |
| 600,600 | (5.691, 5.362)          | (2.836, 2.683) | (19.944, 19.463)       | (8.339, 7.729)   | (82.489, 86.675) | (66.011, 70.527)    | 93  |

**DGP 1.E** In this setup, we focus on type 2 changes with singular  $B_j$ .  $\lambda_{i,0}$  are i.i.d.  $N(0, \frac{1}{r_0}I_{r_0})$  across i and  $\Lambda_0 = (\lambda_{1,0}, \dots, \lambda_{N,0})'$ . In the first regime,  $\Lambda_1 = (\lambda_{1,1}, \dots, \lambda_{N,1})' = \Lambda_0 B_1$  with  $B_1 = [1, 0, 0; 0, 1, 0; 0, 0, 0]$ . In the second regime, we set  $\Lambda_2 = (\lambda_{1,2}, \dots, \lambda_{N,2})' = \Lambda_0 B_2$ , where  $B_2 = [2, x, y; 0, 2, z; 0, 0, 0]$  and x, y, z are i.i.d. N(0, 1). In the third regime,  $\Lambda_3 = (\lambda_{1,3}, \dots, \lambda_{N,3})' = \Lambda_0 B_3$  with  $B_3 = [0, 0, 0; 0, 0, 0; 0, 0, 1]$ . In this setup, we can find a nonsingular matrix **R** such that

$$B_1 \times \mathbf{R} = \begin{bmatrix} 1, \ 0, \ 0\\ 0, \ 1, \ 0\\ 0, \ 0, \ 0 \end{bmatrix} \times \begin{bmatrix} 2, \ x, \ y\\ 0, \ 2, \ z\\ 0, \ 0, \ 1 \end{bmatrix} = \begin{bmatrix} 2, \ x, \ y\\ 0, \ 2, \ z\\ 0, \ 0, \ 0 \end{bmatrix} = B_2.$$

The first breakpoint in this setting corresponds to the case in Type 2 with  $r = 3 > r_{1,2} = r_1 = r_2 = 2$ . As shown in Table 6, the simulation results are similar to DGP 1.D.

Table 6: Simulated root mean squared errors (RMSEs) and mean absolute errors (MAEs) of  $\hat{k}_{QML}$ ,  $\hat{k}_{BKW}$  and  $\hat{k}_{MS}$ , under DGP 1.E. The last column, "Num," represents the number of times the MS method correctly estimated two breakpoints in 1000 iterations.

| N, T    | $(\hat{k}_1, \hat{k}_2)$ | $_2)_{QML}$    | $(\hat{k}_1,\hat{k}_2)_{BKW}$ |                 | $(\hat{k}_1, \hat{k}_2)_{MS}$ |                       |     |
|---------|--------------------------|----------------|-------------------------------|-----------------|-------------------------------|-----------------------|-----|
|         | RMSE                     | MAE            | RMSE                          | MAE             | RMSE                          | MAE                   | Num |
|         |                          |                | $\rho = 0$                    | $\alpha = 0$    | $\beta = 0$                   |                       |     |
| 100,100 | (3.633, 0.155)           | (1.949, 0.022) | (15.145, 4.496)               | (8.667, 1.695)  | (29.270, 5.861)               | (25.831, 3.295)       | 237 |
| 100,300 | (2.520, 0.148)           | (1.366, 0.022) | (8.309, 0.780)                | (3.844, 0.300)  | (48.334, 16.411)              | (36.875, 4.296)       | 311 |
| 300,300 | (2.427, 0.100)           | (1.310, 0.010) | (8.784, 1.627)                | (3.799, 0.452)  | (71.554, 4.018)               | (60.385, 0.718)       | 156 |
| 300,600 | (2.174, 0.078)           | (1.225, 0.006) | (5.545, 0.839)                | (2.842, 0.346)  | (188.222, 8.533)              | (167.173, 3.990)      | 404 |
| 600,600 | (2.477, 0.084)           | (1.290, 0.007) | (8.184, 0.884)                | (3.505, 0.366)  | (212.646, 6.323)              | (207.218, 2.655)      | 174 |
|         |                          |                | $\rho = 0.7$                  | $\alpha = 0$    | $\beta = 0$                   |                       |     |
| 100,100 | (6.912, 0.424)           | (3.832, 0.024) | (25.394, 9.812)               | (19.344, 5.912) | (23.705, 6.482)               | (19.818, 3.608)       | 347 |
| 100,300 | (5.773, 0.118)           | (3.134, 0.014) | (39.502, 12.142)              | (19.971, 3.552) | (38.722, 16.872)              | (31.729, 4.553)       | 425 |
| 300,300 | (6.362, 0.071)           | (3.155, 0.005) | (39.445, 11.937)              | (19.855, 3.580) | (91.617, 6.655)               | (79.851, 3.000)       | 208 |
| 300,600 | (5.176, 0.089)           | (2.636, 0.008) | (29.764, 9.061)               | (11.098, 1.201) | (163.143, 7.993)              | (133.892, 3.420)      | 343 |
| 600,600 | (5.373, 0.055)           | (2.730, 0.003) | (39.830, 10.978)              | (14.478, 1.843) | (224.301, 9.778)              | (218.798, 5.816)      | 163 |
|         |                          |                | $\rho = 0$                    | $\alpha = 0.3$  | $\beta = 0$                   |                       |     |
| 100,100 | (3.913, 0.675)           | (1.964, 0.058) | (15.890, 5.301)               | (9.032, 2.079)  | (26.293, 4.959)               | (22.084, 2.821)       | 308 |
| 100,300 | (2.436, 0.179)           | (1.310, 0.032) | (9.848, 2.290)                | (4.028, 0.376)  | (34.548, 15.027)              | (27.554, 3.916)       | 296 |
| 300,300 | (2.653, 0.100)           | (1.367, 0.010) | (11.124, 2.851)               | (4.060, 0.541)  | (52.027, 3.326)               | (38.841, 1.053)       | 454 |
| 300,600 | (2.409, 0.100)           | (1.344, 0.010) | (5.755, 0.925)                | (3.083, 0.364)  | (67.729, 1.197)               | (55.553, 0.678)       | 528 |
| 600,600 | (2.317, 0.055)           | (1.238, 0.003) | (7.029, 0.908)                | (3.050, 0.370)  | (84.391, 2.551)               | (64.069, 0.907)       | 551 |
|         |                          |                | $\rho = 0$                    | $\alpha = 0$    | $\beta = 0.3$                 |                       |     |
| 100,100 | (3.280, 0.145)           | (1.713, 0.019) | (16.181, 5.220)               | (9.266, 2.009)  | (26.854, 5.717)               | (22.728, 3.244)       | 283 |
| 100,300 | (2.288, 0.145)           | (1.222, 0.019) | (9.004, 2.638)                | (3.879, 0.435)  | (41.812, 18.103)              | (31.784, 5.114)       | 254 |
| 300,300 | (2.540, 0.145)           | (1.276, 0.021) | (12.016, 3.572)               | (4.384, 0.610)  | (77.056, 2.844)               | (65.225, 0.879)       | 182 |
| 300,600 | (2.093, 0.084)           | (1.141, 0.007) | (5.856, 0.896)                | (2.963, 0.350)  | (159.473, 4.668)              | (129.832, 2.168)      | 459 |
| 600,600 | (2.062, 0.071)           | (1.176, 0.005) | (6.939, 0.985)                | (3.254, 0.399)  | (211.883, 6.943)              | $(203.669, \! 3.291)$ | 175 |
|         |                          |                | $\rho = 0.7$                  | $\alpha = 0.3$  | $\beta = 0.3$                 |                       |     |
| 100,100 | (7.263, 1.070)           | (3.980, 0.083) | (25.069, 9.428)               | (19.206, 5.684) | (21.130, 7.205)               | (17.294, 4.071)       | 350 |
| 100,300 | (5.830, 0.182)           | (3.052, 0.027) | (44.287, 14.081)              | (22.931, 4.522) | (38.584, 18.216)              | (32.287, 5.701)       | 442 |
| 300,300 | (6.004, 0.063)           | (2.937, 0.004) | (41.843, 12.271)              | (21.719, 3.710) | (43.257, 3.377)               | (33.133, 1.039)       | 490 |
| 300,600 | (5.743, 0.045)           | (2.893, 0.002) | (34.972, 9.347)               | (13.471, 1.366) | (72.697, 1.046)               | (59.165, 0.464)       | 571 |
| 600,600 | (5.904, 0.000)           | (2.907, 0.000) | (30.896, 9.040)               | (11.606, 1.317) | (73.199, 2.969)               | (59.281, 0.674)       | 570 |

## DGP 2.

Table 7 reports the percentage of correct detection of the number of breaks under DGP 1.A-E using the information criterion (6). From table 7, it can be seen that the information criterion we proposed can almost correctly identify the number of breakpoints as N and T increases.

| $\overline{N,T}$ | DGP 1.A | DGP 1.B $(b=1)$ | DGP 1.B $(b=0)$ | DGP 1.C | DGP 1.D   | DGP 1.E |
|------------------|---------|-----------------|-----------------|---------|-----------|---------|
|                  |         | $\rho = 0$      | $\alpha = 0$    |         | β =       | = 0     |
| 100,100          | 1.000   | 0.725           | 0.488           | 0.991   | 0.331     | 0.422   |
| 100,300          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 300,300          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 0.999   |
| 300,600          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 600,600          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
|                  |         | $\rho = 0.7$    | $\alpha = 0$    |         | $\beta$ = | = 0     |
| 100,100          | 1.000   | 0.110           | 0.032           | 0.970   | 0.075     | 0.109   |
| 100,300          | 0.999   | 1.000           | 1.000           | 0.999   | 0.990     | 0.956   |
| 300,300          | 1.000   | 1.000           | 1.000           | 1.000   | 0.940     | 0.890   |
| $300,\!600$      | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 600,600          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 0.999   |
|                  |         | $\rho = 0$      | $\alpha = 0.3$  |         | $\beta$ = | = 0     |
| 100,100          | 1.000   | 0.634           | 0.451           | 0.971   | 0.304     | 0.360   |
| 100,300          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 300,300          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 0.999   |
| $300,\!600$      | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 600,600          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
|                  |         | $\rho = 0$      | $\alpha = 0$    |         | $\beta =$ | 0.3     |
| 100,100          | 1.000   | 0.644           | 0.391           | 0.995   | 0.321     | 0.376   |
| 100,300          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 300,300          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 0.999   |
| 300,600          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 600,600          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
|                  |         | $\rho = 0.7$    | $\alpha = 0.3$  |         | $\beta =$ | 0.3     |
| 100,100          | 1.000   | 0.052           | 0.012           | 0.956   | 0.069     | 0.122   |
| 100,300          | 1.000   | 1.000           | 1.000           | 1.000   | 0.984     | 0.958   |
| $300,\!300$      | 1.000   | 1.000           | 1.000           | 1.000   | 0.930     | 0.899   |
| $300,\!600$      | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 1.000   |
| 600,600          | 1.000   | 1.000           | 1.000           | 1.000   | 1.000     | 0.999   |

Table 7: Percentage of correct detection of the number of breaks under DGP 1.A, DGP 1.B, DGP 1.C, DGP 1.D, DGP 1.E

#### DGP 3.

Finally, we consider the accuracy of the information criterion (6) in choosing the number of breakpoints when the the number of breaks from 0 to 4. We let the factor loadings within each regime,  $\lambda_i$ ,  $i = 1 \cdots, N$ , follow a normal distribution  $N(0, 1/r_0)$ , which means that the factor loadings between regimes are independent. Before estimating the factors and factor loadings, we use the IC criteria from Bai and Ng (2002) to determine the number of factors for the full samples. To determine whether a breakpoint exists, we compare the size of the loss function and the penalty function when considering a single breakpoint. In the absence of a breakpoint, the loss function is zero. Thus, we need to compare the divergence rates of the loss function and the penalty function when overfitting with one breakpoint. If the divergence rate of the loss function is less than that of the penalty function, it indicates that no breakpoint is present; otherwise, a breakpoint exists. From Table 8, it can be seen that as the sample size increases, the probability that the proposed information criterion accurately selects the number of breakpoints asymptotically converges to 1. This also validates our theorem 3.

| N, T        | 0                                   | 1             | 2                          | 3     | 4     |  |  |  |
|-------------|-------------------------------------|---------------|----------------------------|-------|-------|--|--|--|
|             | ho=0, lpha=0, eta=0                 |               |                            |       |       |  |  |  |
| 100,300     | 1.000                               | 1.000         | 1.000                      | 1.000 | 0.934 |  |  |  |
| 300,300     | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 100,600     | 0.980                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 300,600     | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 600,600     | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
|             |                                     | $\rho = 0.$   | $7, \alpha = 0, \beta = 0$ |       |       |  |  |  |
| 100,300     | 0.920                               | 1.000         | 1.000                      | 1.000 | 0.082 |  |  |  |
| 300,300     | 0.980                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 100,600     | 0.880                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| $300,\!600$ | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 600,600     | 0.960                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
|             | $\rho = 0, \alpha = 0.3, \beta = 0$ |               |                            |       |       |  |  |  |
| 100,300     | 1.000                               | 1.000         | 1.000                      | 1.000 | 0.828 |  |  |  |
| $300,\!300$ | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| $100,\!600$ | 0.980                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| $300,\!600$ | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 600,600     | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
|             |                                     | $\rho = 0,$   | $\alpha=0,\beta=0.3$       |       |       |  |  |  |
| 100,300     | 1.000                               | 1.000         | 1.000                      | 1.000 | 0.903 |  |  |  |
| $300,\!300$ | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| $100,\!600$ | 0.980                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| $300,\!600$ | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 600,600     | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
|             |                                     | $\rho = 0.7,$ | $\alpha=0.3,\beta=0.3$     |       |       |  |  |  |
| 100,300     | 0.920                               | 1.000         | 1.000                      | 1.000 | 0.033 |  |  |  |
| 300,300     | 0.980                               | 1.000         | 1.000                      | 1.000 | 0.451 |  |  |  |
| $100,\!600$ | 0.980                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| $300,\!600$ | 0.940                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |
| 600,600     | 1.000                               | 1.000         | 1.000                      | 1.000 | 1.000 |  |  |  |

Table 8: Percentage of correct detection of the number of breaks under different number of breaks.

# 6 Empirical application

In this section we apply our method to detect break points for the FRED-MD (Federal Reserve Economic Data - Monthly Data) data set. The FRED database is maintained by

the Research division of the Federal Reserve Bank of St. Louis, and is publicly accessible and updated in real-time. The August 2024 vintage of the FRED-MD dataset includes 126 unbalanced monthly time series spanning from January 1959 to July 2024. These series are categorized into eight groups: (1) output and income, (2) labor market, (3) housing, (4) consumption, orders, and inventories, (5) money and credit, (6) interest rates and exchange rates, (7) prices, and (8) stock market. Following the literature, we replaced the missing values replaced by the EM algorithm and transform the data by removing first two months and replaced outliers, so that we obtain a total of T = 785 monthly observations for each macroeconomic variable. The data have been centered and standardized for the analysis. One can refer to McCracken and Ng (2016) for the detailed data description.

We use h = 20 as the minimum distance between breakpoints. Although theoretically, the distance between breakpoints is required to be proportional to T, in practical applications, a smaller h can be chosen. This is because a smaller h often allows for the detection of more breakpoints, even when the structural changes are minor. We select the number of factors by the information criteria IC2 of Bai and Ng (2002) for the entire samples. As a result, the number of selected factors is 7. Next, we apply our proposed information criterion (6) and the QML method to select the number of breakpoints and estimate their locations as follows: January 1969, January 1983, June 2008, March 2010, and February 2020. By January 1969, U.S. inflation surged due to heightened spending on the Vietnam War and "Great Society" programs. By January 1983, the economy was recovering from a severe recession triggered by the Federal Reserve's anti-inflationary policies, though unemployment remained high. June 2008 marked the early stages of the Global Financial Crisis, as the collapse of the housing market spread into the broader financial sector. By March 2010, the U.S. economy was slowly emerging from the Great Recession, bolstered by fiscal stimulus and the passage of the Affordable Care Act. February 2020 saw the initial economic disruptions from the COVID-19 pandemic, which would soon result in a sharp global downturn.

Finally, based on the five breakpoints, we divided the entire period into six regimes and applied the IC2 criterion from Bai and Ng (2002) to estimate the number of factors in each regime, yielding 2, 5, 7, 3, 4, and 7, respectively. The variation in the number of factors

between adjacent breakpoints indicates clear changes. Therefore, in accordance with theorem 2, we conclude that each estimated breakpoint is consistent.

# 7 Conclusions

In this paper, we examine the quasi-maximum likelihood (QML) method for estimating breakpoints in high-dimensional factor models with multiple structural changes. We study two types of changes and develop asymptotic theory for the QML estimators. For type 1 changes, we establish the convergence rate between the eigenvalues of the estimated factor covariance matrix and those of the sub-regime covariance matrix, demonstrating the consistency of the QML estimators. For type 2 changes, we show that the distance between the QML estimators and the true break points is bounded. The proposed method is straightforward to implement, computationally efficient, and theoretically robust, as it requires performing eigendecomposition only once. Additionally, we also show that the number of changes can be consistently estimated via information criterion approach. Simulation results confirm the strong finite-sample performance of our approach. Finally, applying the method to the FRED-MD dataset further demonstrates its practical utility.

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