

ON THE INITIAL IDEAL OF A GENERIC ARTINIAN GORENSTEIN ALGEBRA

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ABSTRACT. In this note we show that the initial ideal of the annihilator ideal of a generic form is generated by the largest possible monomials in each degree. We also show that the initial ideal with respect to the degree reverse lexicographical ordering of the annihilator ideal of the complete symmetric form has this property, by determining a minimal Gröbner basis of it. Moreover, we determine the total Betti numbers for a class of strongly stable monomial ideals and show that these numbers agree with those for the degree reverse lexicographical initial ideals of the ideal generated by a sufficiently large number of generic forms, and of the annihilator ideal of a generic form.

1. INTRODUCTION

Let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring. degree reverse lexicographical ordering. Moreno-Socías [8] conjectured that the initial ideal I of an ideal generated by m generic forms is *weakly reverse lexicographical*, which means that if $x^\mu \in I$ is a minimal monomial generator of I , then every monomial of the same degree as x^μ that precedes x^μ with respect to the degree reverse lexicographical ordering is also in I . Pardue [9] showed that Moreno-Socías' conjecture for the special case $m = n$ implies the longstanding Fröberg conjecture [4] on the Hilbert series of ideals generated by any number of generic forms. We also recall that a monomial ideal I is *reverse lexicographic* if, for every $d \geq 0$, I_d is generated by the largest possible monomials of degree d with respect to the reverse lexicographic ordering.

Let $S = k[X_1, \dots, X_n]$. We view S (the inverse system of R) as an R -module, with the action of R on S being given by $x_i \circ g = \frac{\partial}{\partial x_i} g$ for every $g \in S$ and $i = 1, \dots, n$. If A is a subset of S we set

$$\text{Ann}(A) = \{f \in R : f \circ g = 0 \text{ for all } g \in A\}.$$

In the case of positive characteristic, we have to replace the *differentiation action* of S on R defined above by the *contraction action*, given by

$$x_i \circ X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n} = \begin{cases} X_1^{a_1} X_2^{a_2} \cdots X_i^{a_i-1} \cdots X_n^{a_n} & \text{if } a_i \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let G be a generic form of degree e in S . Then the Gorenstein algebra $R/\text{Ann}(G)$ is compressed [7], i.e., the Hilbert series is equal to

$$\sum_{i=0}^{\lfloor e/2 \rfloor} \dim_k R_i t^i + \sum_{i=\lfloor e/2 \rfloor+1}^e \dim_k R_{e-i} t^i.$$

Much is known about the generators of $\text{Ann}(G)$. If e is even, then $\text{Ann}(G)$ is generated in degree $e/2+1$ (this in fact holds for every compressed Artinian algebra of even socle degree e [6, Example 4.7]). For e odd, $\text{Ann}(G)$ is generated in degree $\lfloor e/2 \rfloor + 1$ if $n \geq 4$ or if $n = 3$ and $\lfloor e/2 \rfloor$ is odd ([2, Corollary 4.5]). In the case where $n = 3$, e is odd and $\lfloor e/2 \rfloor$ is even, then $\text{Ann}(G)$ has generators in degree $\lfloor e/2 \rfloor + 1$ and at least one generator in degree $\lfloor e/2 \rfloor + 2$ ([1, Example 3.16]).

In our main result of this manuscript, we focus on determining the initial ideal of $\text{Ann}(G)$ when $G \in S$ is a generic form of degree e . For any monomial order on R we show that $\text{in}(\text{Ann}(G))$ is generated by the largest possible monomials in each degree. If we consider the degree reverse lexicographic order, then this shows that the ideal is not only weakly reverse lexicographic, but is in fact reverse lexicographic, thus providing a connection to the currently open Moreno-Socías' conjecture.

Let H_e be the complete symmetric homogeneous form of degree e in S . The first author, Migliore, Miró-Roig, and Nagel [3] showed that $R/\text{Ann}(G)$ and $R/\text{Ann}(H_e)$ have the same Hilbert series, so that in particular, $R/\text{Ann}(H_e)$ is also compressed. This was achieved by determining the generators for $\text{Ann}(H_e)$. They also showed that $R/\text{Ann}(H_e)$ has the strong Lefschetz property by showing that the sum of the variables serves as a strong Lefschetz element. Here, we extend this result by determining a reduced Gröbner basis of $\text{Ann}(H_e)$ with respect to the degree reverse lexicographical ordering. As a consequence, we obtain that $\text{in}(\text{Ann}(H_e)) = \text{in}(\text{Ann}(G))$, from which we get that the initial ideal of $\text{Ann}(H_e)$ is reverse lexicographical and that x_n serves as a strong Lefschetz element.

Let $d = \lfloor e/2 \rfloor + 1$. The ideal $\text{Ann}(G)$ has at least $\dim_k R_d - \dim_k R_{d-1}$ minimal generators of degree d . Consider the ideal I generated by at least as many general forms of degree d , and possibly more of degrees $> d$. We provide a connection between the initial ideal of I and the initial ideal of $\text{Ann}(G)$, with respect to the degree reverse lexicographical ordering, by showing that they have the same total Betti numbers. This relationship is unexpected since R/I and $R/\text{Ann}(G)$ are not even isomorphic as vector spaces.

The paper is organized as follows. In Section 2 we determine the initial ideal of the annihilator ideal of a generic form. In Section 3 we determine the Gröbner basis of the annihilator ideal of the complete symmetric homogeneous form. In Section 4 we determine homological invariants of a class of strongly stable monomial ideals, of the initial ideal of the annihilator ideal of a generic form, and of the initial ideal of the ideal generated by sufficiently many generic forms.

2. THE GENERIC CASE

Theorem 2.1. *Let $G \in S$ be a generic homogeneous form of degree e and let \prec be any monomial order. Then for $d \in [e/2, e]$*

$$[\text{in}(\text{Ann}(G))]_d = \langle \text{first } \dim_k R_d - \dim_k R_{e-d} \text{ monomials of degree } d \rangle.$$

Proof. We compute the degree d part of the ideal $\text{Ann}(G)$ as the kernel of the catalecticant matrix $\text{Cat}_{e-d}^d(G)$ representing the pairing

$$S_d \times S_{e-d} \longrightarrow k$$

given by $(f, g) \mapsto (fg) \circ G$. Let \mathcal{M}_d be the monomial basis for R_d . We can write $G = \sum_{\mu \in \mathcal{M}_e} \frac{1}{\mu \circ \mu} a_\mu \mu$. Observe that $\mu \circ \mu = 1$ for all monomials μ if we are in characteristic p using the contraction action. The catalecticant matrix for

computing $\text{Ann}(G)$ in degree d is given by

$$(a_{\mu\nu})_{\mu \in \mathcal{M}_{e-d}, \nu \in \mathcal{M}_d}$$

Let $S \subseteq \mathcal{M}_d$ and $T \subseteq \mathcal{M}_{e-d}$ be subsets of the same cardinality and consider the minor defined by them. If we order the rows and columns according to our monomial order, starting with the largest, we have that the indices of the entries of the catalecticant matrix is decreasing in rows and columns. Hence, the product of the diagonal elements in the minor is a squarefree monomial in the coefficients a_μ that cannot occur in any other term of the minor. This shows that any minor is non-zero for a generic form. This means that we can row-reduce the catalecticant matrix to a form

$$\begin{bmatrix} B & I \end{bmatrix}$$

and the kernel is given by the columns of

$$\begin{bmatrix} I \\ -B \end{bmatrix}$$

showing that the initial monomials of the basis of $[\text{Ann}(G)]_d$ are the first monomials in the monomial order that we have chosen. \square

We now specify further properties of the ideal $\text{in}(\text{Ann}(G))$ when we consider on R the degree reverse lexicographic order induced by $x_1 \succ \cdots \succ x_n$.

Corollary 2.2. *Let $G \in S$ be a generic form of degree e . Consider on R the degree reverse lexicographic order. Then x_n is a strong Lefschetz element for $R/\text{in}(\text{Ann}(G))$.*

Proof. Let $J = \text{in}(\text{Ann}(G))$ and $B = R/J$. For x_n to be a strong Lefschetz element for B what is required is that $\cdot x_n^{e-2i} : B_i \rightarrow B_{e-i}$ is bijective for all $i < e/2$. But for such values of i , $B_i = R_i$ since B is compressed. Also, by Theorem 2.1, a monomial generating set of B_{e-i} consists of the smallest $\dim_k R_i$ monomials of degree $e - i$. These are precisely all monomials of degree $e - i$ that are divisible by x_n^{e-2i} . We conclude that $\cdot x_n^{e-2i} : B_i \rightarrow B_{e-i}$ is bijective for all $i < e/2$. \square

For each $i \geq 0$ we will denote by \mathcal{A}_i the monomial basis of $k[x_1, \dots, x_{n-1}]_i$.

Corollary 2.3. *Let $G \in S$ be a generic form of degree e . Consider on R the degree reverse lexicographic order.*

If $e = 2d + 1$ is odd, then the minimal monomial basis of $\text{in}(\text{Ann}(G))$ in degree $d + 1 + i$ consists precisely of the monomials in $x^{2i}\mathcal{A}_{d+1-i}$ for each $0 \leq i \leq d + 1$.

If $e = 2d$ is even, then the minimal monomial basis of $\text{in}(\text{Ann}(G))$ in degree $d + 1 + i$ consists precisely of the monomials in $\mathcal{A}_{d+1} \cup x_n \mathcal{A}_d$ for $i = 0$ and of the monomials in $x^{2i+1}\mathcal{A}_{d-i}$ for each $1 \leq i \leq d$.

Proof. $J = \text{in}(\text{Ann}(G))$ is generated in degrees $\lfloor e/2 \rfloor + 1 = d + 1 \leq j \leq e + 1$. By Theorem 2.1, J is minimally generated in degree $d + 1$ by the first $\dim_k R_{d+1} - \dim_k R_{e-d-1}$ monomials of degree $d + 1$. In the case $e = 2d + 1$ these consist of the monomials in \mathcal{A}_{d+1} and in the case $e = 2d$ these consist of the monomials in $\mathcal{A}_{d+1} \cup x_n \mathcal{A}_d$.

For each $j > d + 1$, by Theorem 2.1, J_{j-1} is generated by all monomials of degree $j - 1$ not divisible by $x_n^{2(j-1)-e}$, so that $R_1 J_{j-1}$ is generated by all monomials of degree j not divisible by x_n^{2j-1-e} . Therefore, the minimal generators of J of degree j are precisely the monomials of degree j that are divisible by x_n^{2j-1-e} and not by

x_n^{2j-e} , i.e., the monomials in $x_n^{2j-1-e}\mathcal{A}_{e-j+1}$. Putting $j = d + 1 + i$ with $i \geq 1$ we obtain the desired statement. \square

3. A GRÖBNER BASIS FOR THE ANNIHILATOR OF THE COMPLETE HOMOGENEOUS POLYNOMIAL

In this section we assume that the field k has characteristic zero. We also fix on $R = k[x_1, \dots, x_n]$ the degree reserve lexicographic order induced by $x_1 \succ \dots \succ x_n$.

For each $e \geq 0$ denote by

$$H_e = \sum_{\mathbf{i} \in \mathbb{N}^n, |\mathbf{i}|=e} X_1^{i_1} \cdots X_n^{i_n} \in S$$

the *complete symmetric polynomial* of degree e . In [3] the authors have shown that $R/\text{Ann}(H_e)$ is compressed ([3, Theorem 2.11]). Thus, $R/\text{Ann}(H_e)$ and $R/\text{Ann}(G)$ have the same Hilbert function, where $G \in S$ is a generic form of degree e . In their paper, the authors also determine a set of generators of $\text{Ann}(H_e)$ ([3, Theorem 2.12]) and show that $R/\text{Ann}(H_e)$ has the strong Lefschetz Property.

In what follows, we focus on determining $\text{in}(\text{Ann}(H_e))$. We will show that $\text{in}(\text{Ann}(H_e)) = \text{in}(\text{Ann}(G))$, from which it follows that $\text{in}(\text{Ann}(H_e))$ is reverse lexicographical. We achieve this by specifying a Gröbner basis for $\text{in}(\text{Ann}(H_e))$. In order to present such a basis, we introduce some notation.

Definition 3.1. Let $\varphi : R \rightarrow R$ be the k -linear map given by

$$\varphi(x_1^{i_1} \cdots x_n^{i_n}) = \frac{1}{i_1! \cdots i_n!} x_1^{i_1} \cdots x_n^{i_n}$$

for each $(i_1, \dots, i_n) \in \mathbb{N}^n$.

Definition 3.2. For each $\underline{a} = (a_1, \dots, a_{n-1}) \in \mathbb{N}^{n-1}$ denote

$$F_{\underline{a}} = (x_1 - x_n)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}} \in R.$$

For every $d \geq 1$, in [3, Theorem 2.12] the authors have shown that $F_{\underline{a}}$, with $|\underline{a}| = d + 1$, annihilates H_{2d} and H_{2d+1} . We now show the following

Proposition 3.3. If $e = 2d$ is even then $\{\varphi(x_n^{2i+1}F_{\underline{a}}) : i = 0, \dots, d \text{ and } |\underline{a}| = d - i\} \subset \text{Ann}(H_{2d})$ and if $e = 2d + 1$ is odd then $\{\varphi(x_n^{2i}F_{\underline{a}}) : i = 0, \dots, d + 1 \text{ and } |\underline{a}| = d + 1 - i\} \subset \text{Ann}(H_{2d+1})$.

Proof. We do induction on e . For $e = 1$ the statement amounts to verifying that $x_i - x_n$, for $i = 1, \dots, n - 1$, as well as $\frac{1}{2}x_n^2$, annihilate $h_1 = X_1 + \dots + X_n$, which is obvious.

Suppose now that $e = 2d$ is even and that the statement is true for all $e_0 < 2d$. Let $i \in \{0, \dots, d\}$ and $\underline{a} = (a_1, \dots, a_{n-1})$ be such that $|\underline{a}| + i = d$. We need to show that $\varphi(x_n^{2i+1}F_{\underline{a}}) \circ H_{2d} = 0$.

By [3, Theorem 2.12], considering $n + 1$ variables x_0, x_1, \dots, x_n , we have that

$$f = \varphi'((x_1 - x_n)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}}(x_0 - x_n)^{2i+1}) = \varphi'(F_{\underline{a}}(x_0 - x_n)^{2i+1})$$

annihilates $H'_{2(|\underline{a}|+2i+1)-1} = H'_{2d+2i+1}$. Here φ' stands for the map φ defined on $k[x_0, x_1, \dots, x_n]$ instead of $k[x_1, \dots, x_n]$ and, for any e' , $H'_{e'}$ stands for the complete

symmetric polynomial of degree e' in the variables X_0, X_1, \dots, X_n . We observe that

$$\begin{aligned} f &= \varphi'(F_{\underline{a}}(x_0 - x_n)^{2i+1}) = \varphi' \left(\sum_{j=0}^{2i+1} \binom{2i+1}{j} (-1)^j x_n^j x_0^{2i+1-j} F_{\underline{a}} \right) \\ &= \sum_{j=0}^{2i+1} \binom{2i+1}{j} \frac{(-1)^j x_0^{2i+1-j}}{(2i+1-j)!} \varphi(x_n^j F_{\underline{a}}) \\ &= \sum_{j=0}^{2i+1} b_j x_0^{2i+1-j} \varphi(x_n^j F_{\underline{a}}). \end{aligned}$$

Here the b_j represent the suitable coefficients. The third equality holds because x_0 does not show up in $F_{\underline{a}}$. Since

$$H'_{2d+2i+1} = \sum_{t=0}^{2d+2i+1} X_0^t H_{2d+2i+1-t}$$

and since

$$(b_j x_0^{2i+1-j} \varphi(x_n^j F_{\underline{a}})) \circ (X_0^t H_{2d+2i+1-t}) = b'_j X_0^{t-(2i+1-j)} (\varphi(x_n^j F_{\underline{a}}) \circ H_{2d+2i+1-t})$$

for suitable coefficients b'_j , the coefficient of X_0^{2i+1} in $f \circ H'_{2d+2i+1}$ is

$$c = \sum b'_j (\varphi(x_n^j F_{\underline{a}}) \circ H_{2d+2i+1-t})$$

where the sum ranges over all j, t such that $0 \leq t \leq 2d+2i+1$, $0 \leq j \leq 2i+1$ and $t - (2i+1-j) = 2i+1$. Since we must have $t \geq 2i+1$, we can write $t = 2i+1+r$. Then $j = 2i+1-r$ and we can rewrite the coefficient of X_0^{2i+1} in $f \circ H'_{2d+2i+1}$ as

$$c = \sum_{r=0}^{2i+1} b'_{2i+1-r} (\varphi(x_n^{2i+1-r} F_{\underline{a}}) \circ H_{2d-r})$$

By the induction hypothesis, whenever $r \geq 1$ we have that $\varphi(x_n^{2i+1-r} F_{\underline{a}}) \circ H_{2d-r} = 0$. Therefore

$$c = b'_{2i+1} \varphi(x_n^{2i+1} F_{\underline{a}}) \circ H_{2d}.$$

Since f annihilates $H'_{2d+2i+1}$, we must have $c = 0$, i.e., $\varphi(x_n^{2i+1} F_{\underline{a}}) \circ H_{2d} = 0$, as we wanted to show.

Suppose now that $e = 2d+1$ is odd and that the statement is true for all $e_0 < 2d+1$. In this case we need to show that $\varphi(x_n^{2i} F_{\underline{a}}) \circ H_{2d+1} = 0$ for every $i \in \{1, \dots, d+1\}$ and $\underline{a} = (a_1, \dots, a_{n-1})$ such that $|a| + i = d+1$. The proof follows along the same lines as in the case of e even, except this time one starts the argument from the fact that

$$f = \varphi'((x_1 - x_n)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}} (x_0 - x_n)^{2i}) = \varphi'(F_{\underline{a}}(x_0 - x_n)^{2i})$$

annihilates $H'_{2(|a|+2i-1)} = H'_{2d+2i}$. We obtain, as before, that the coefficient of X_0^{2i-1} in $f \circ H'_{2d+2i}$ is

$$c = \sum_{r=0}^{2i} b'_{2i-r} (\varphi(x_n^{2i-r} F_{\underline{a}}) \circ H_{2d+1-r})$$

for suitable nonzero coefficients b'_j . By the induction hypothesis, whenever $r \geq 1$ we have that $\varphi(x_n^{2i-r} F_{\underline{a}}) \circ H_{2d+1-r} = 0$. Therefore

$$c = b'_{2i} \varphi(x_n^{2i} F_{\underline{a}}) \circ H_{2d+1}.$$

Since f annihilates H'_{2d+2i} , we must have $c = 0$, i.e., $\varphi(x_n^{2i} F_{\underline{a}}) \circ H_{2d+1} = 0$. \square

We can now specify a minimal Gröbner basis for $\text{Ann}(H_e)$. We recall that we are considering on R the degree reverse lexicographic ordering.

Theorem 3.4. *For any $e \geq 1$, if $G \in S$ is a generic form of degree e then*

$$\text{in}(\text{Ann}(G)) = \text{in}(\text{Ann}(H_e)).$$

Moreover, if $e = 2d$ is even then

$$\mathcal{G}_{2d} = \{\varphi(F_{\underline{a}}) : |\underline{a}| = d + 1\} \cup \{\varphi(x_n^{2i+1} F_{\underline{a}}) : i = 0, \dots, d \text{ and } |\underline{a}| = d - i\}$$

is a minimal Gröbner basis for $\text{Ann}(H_{2d})$. If $e = 2d + 1$ is odd then

$$\mathcal{G}_{2d+1} = \{\varphi(x_n^{2i} F_{\underline{a}}) : i = 0, \dots, d + 1 \text{ and } |\underline{a}| = d + 1 - i\}$$

is a minimal Gröbner basis for $\text{Ann}(H_{2d+1})$.

Proof. Suppose $e = 2d + 1$ is odd. By Proposition 3.3 and [3, Theorem 2.12], \mathcal{G}_{2d+1} is a subset of $\text{Ann}(H_e)$. The initial terms of \mathcal{G}_{2d+1} form the set $\{x_n^{2i} x_1^{a_1} \cdots x_{n-1}^{a_{n-1}} : i = 0, \dots, d + 1 \text{ and } a_1 + \cdots + a_{n-1} = d + 1 - i\}$, which, by Corollary 2.3, is exactly the minimal monomial generating set of $\text{in}(\text{Ann}(H_{2d+1}))$. Since $R/\text{in}(\text{Ann}(H_{2d+1}))$ and $R/\text{in}(\text{Ann}(G))$ have the same Hilbert function, we conclude that $\text{in}(\text{Ann}(G)) = \text{in}(\text{Ann}(H_{2d+1}))$ and that \mathcal{G}_{2d+1} is a minimal Gröbner basis for $\text{Ann}(H_{2d+1})$. For $e = 2d$ even, the result follows in the same way. \square

We finish this section with a couple of observations concerning the reducedness of the Gröbner basis presented in Theorem 3.4 and concerning the problem of determining a minimal generating set for $\text{Ann}(H_e)$.

Remark 3.5. It is easy to see that the Gröbner basis \mathcal{G}_{2d+1} for $\text{Ann}(H_{2d+1})$ presented in Theorem 3.4 is reduced. However, the Gröbner basis \mathcal{G}_{2d} for $\text{Ann}(H_{2d})$ is not in reduced form. This is because, given $(a_1, \dots, a_{n-1}) \in \mathbb{N}^{n-1}$ with $a_1 + \cdots + a_{n-1} = d + 1$, we have that

$$\begin{aligned} \varphi(F_{(a_1, \dots, a_{n-1})}) &= \varphi((x_1 - x_n)^{a_1} \cdots (x_{n-1} - x_n)^{a_{n-1}}) \\ &= \frac{x_1^{a_1} \cdots x_{n-1}^{a_{n-1}}}{a_1! \cdots a_{n-1}!} - x_n \left(\sum_{i=1}^{n-1} a_i \frac{x_1^{a_1} \cdots x_i^{a_i-1} \cdots x_{n-1}^{a_{n-1}}}{a_1! \cdots (a_i - 1)! \cdots a_{n-1}!} \right) + \cdots \end{aligned}$$

has in its support monomials divisible by the leading monomials of the Gröbner basis elements $\varphi(x_n F_{(a_1, \dots, a_i-1, \dots, a_{n-1})})$ with $a_i \neq 0$. As such, replacing in \mathcal{G}_{2d} each $\varphi(F_{(a_1, \dots, a_{n-1})})$ with $a_1 + \cdots + a_{n-1} = d + 1$ by

$$\varphi(F_{(a_1, \dots, a_{n-1})}) + \sum_{i=1}^{n-1} a_i \varphi(x_n F_{(a_1, \dots, a_i-1, \dots, a_{n-1})})$$

we obtain a reduced Gröbner basis of $\text{Ann}(H_{2d})$.

Remark 3.6. According to [3, Theorem 2.12], for every $d \geq 0$, $\text{Ann}(H_{2d})$ is equigenerated in degree $d + 1$, while $\text{Ann}(H_{2d+1})$ is generated in degrees $d + 1$ and $d + 2$. Therefore, a minimal set of generators of $\text{Ann}(H_{2d})$ is given by the elements of its minimal Gröbner basis \mathcal{G}_{2d} that have degree $d + 1$, i.e., by $\{\varphi(F_{\underline{a}}) : |\underline{a}| = d + 1\} \cup \{\varphi(x_n F_{\underline{a}}) : |\underline{a}| = d\}$. Similarly, the elements in $\{\varphi(F_{\underline{a}}) : |\underline{a}| = d + 1\}$ are minimal generators of $\text{Ann}(H_{2d+1})$. To determine a minimal set of generators of $\text{Ann}(H_{2d+1})$ it remains to check which elements in $\{\varphi(x_n^2 F_{\underline{a}}) : |\underline{a}| = d\}$ are also minimal generators. We do not have an answer to this. However, our computational experiments point to the following conjecture.

Conjecture 3.7. If d is odd, then $\text{Ann}(H_{2d+1})$ has no minimal generators in degree $d + 2$, so that $\text{Ann}(H_{2d+1}) = (\varphi(F_{\underline{a}}) : |\underline{a}| = d + 1)$.

If d is even, then $\text{Ann}(H_{2d+1})$ has exactly one minimal generator in degree $d + 2$. In fact, we have $\text{Ann}(H_{2d+1}) = (\varphi(F_{\underline{a}}) : |\underline{a}| = d + 1) + (\varphi(x_n^2(x_{n-1} - x_n)^d))$.

4. TOTAL BETTI NUMBERS FOR A CLASS OF STRONGLY STABLE IDEALS

Recall that a monomial ideal J of R is *strongly stable* if for every monomial $m \in J$, every variable x_i that divides m and every $j < i$ the monomial $(x_j/x_i)m$ also belongs to J . In particular, every weakly reverse lexicographic ideal is strongly stable.

In this section we determine the total Betti numbers β_p of strongly stable monomial ideals generated in degrees $\geq d$ that contain every monomial of degree d not divisible by x_n . We then show that β_p equals the total Betti numbers of the initial ideal of an ideal generated by generic forms in degrees $\geq d$ with a sufficient number of generators of degree d , and likewise, that β_p equals the total Betti numbers of the initial ideal of the annihilator of a generic form.

In the following statement, \mathfrak{m} stands for (x_1, \dots, x_n) , the maximal homogeneous ideal of $R = k[x_1, \dots, x_n]$.

Theorem 4.1. *Let $d \geq 2$. Let J be a strongly stable monomial ideal of $R = k[x_1, \dots, x_n]$ generated in degrees $\geq d$ and containing every monomial of degree d not divisible by x_n . Suppose R/J is Artinian. Then*

$$\beta_p(R/J) = \beta_p(R/\mathfrak{m}^d) = \sum_{i=1}^n \binom{d+i-2}{d-1} \binom{i-1}{p-1} \quad \text{for all } p \geq 1.$$

In particular, J is minimally generated by $\dim_k R_d$ monomials.

Proof. We start by proving the last statement, i.e., that J is minimally generated by $\dim_k R_d$ monomials. For each $i \geq 0$ let r_i denote the number of minimal generators of J of degree i . We claim that

$$\text{HF}(R/J, i) - \text{HF}(R/J, i + 1) = r_{i+1}$$

for every $i \geq d$. Since

$$\begin{aligned} \text{HF}(R/J, i + 1) &= \dim_k R_{i+1} - \dim_k J_{i+1} \\ &= \dim_k R_{i+1} - \dim_k(R_1 J_i) - r_{i+1} \\ &= \dim_k(R_{i+1}/R_1 J_i) - r_{i+1}, \end{aligned}$$

the claim is equivalent to showing that

$$\dim_k(R_i/J_i) = \dim_k(R_{i+1}/R_1 J_i).$$

For this, we observe that multiplication by x_n induces a bijection between the monomial basis of R_i/J_i and the monomial basis of R_{i+1}/R_1J_i . The surjectiveness comes from the fact that J_i contains every monomial of degree i not divisible by x_n , thus forcing every element of R_{i+1}/R_1J_i to be a multiple of x_n . To show injectiveness, suppose that m is a monomial of degree i such that $x_n m = x_t m'$ where $1 \leq t \leq n$ and m' is a monomial in J_i . If $t = n$ then $m = m' \in J_i$ and we are done. If $t < n$ then m' must be divisible by x_n and we get $m = m'(x_t/x_n)$, which, since J is strongly stable, belongs to J_i . This proves the claim.

Let s be the socle degree of R/J . Since J is generated in degrees greater or equal to d , we have

$$\begin{aligned} \dim_k R_d - r_d &= \text{HF}(R/J, d) \\ &= \sum_{i=d}^s \text{HF}(R/J, i) - \text{HF}(R/J, i+1) \\ &= \sum_{i=d}^s r_{i+1}. \end{aligned}$$

We conclude that $r_d + \dots + r_{s+1} = \dim_k R_d$, i.e., J is minimally generated by $\dim_k R_d$ monomials.

Consequently, J is minimally generated by all $\dim_k R_d - \dim_k R_{d-1}$ monomials of degree d that are not divisible by x_n and by $\dim_k R_{d-1}$ monomials divisible by x_n . More precisely, for each $1 \leq i < n$, J contains exactly $a_i = \dim_k k[x_1, \dots, x_i]_{d-1} = \binom{d+i-2}{d-1}$ minimal generators that are divisible by x_i and not by any x_j with $j > i$. Also J contains exactly $a_n = \dim_k R_{d-1} = \binom{d+n-2}{d-1}$ minimal generators divisible by x_n .

As J is strongly stable, the minimal free resolution of R/J is given by the Eliahou-Kervaire resolution. According to [10, Corollary 28.12], the total Betti numbers $\beta_p(R/J)$ are given by the formula

$$\beta_p(R/J) = \sum_{i=1}^n a_i \binom{i-1}{p-1} = \sum_{i=1}^n \binom{d+i-2}{d-1} \binom{i-1}{p-1}$$

for all $p \geq 1$. Since \mathfrak{m}^d satisfies the hypothesis of the theorem, the total Betti numbers of R/\mathfrak{m}^d also agree with this formula. \square

As a corollary, we obtain that if I is an ideal of R generated by generic forms of degrees $\geq d$, of which a sufficient number of them have degree exactly d , then adding more generic forms of degrees $\geq d$ to I will not change the total Betti numbers of $R/\text{in}(I)$. In particular, it will not change the number of minimal generators of $\text{in}(I)$.

Corollary 4.2. *Let $d \geq 2$ and $n \geq 3$. Let I be an ideal of $R = k[x_1, \dots, x_n]$ generated by generic forms of degrees greater than or equal to d , of which at least $\dim_k R_d - \dim_k R_{d-1}$ have degree d . Then*

$$\beta_p(R/\text{in}(I)) = \beta_p(R/\mathfrak{m}^d) = \sum_{i=1}^n \binom{d+i-2}{d-1} \binom{i-1}{p-1} \quad \text{for all } p \geq 1.$$

In particular, $\text{in}(I)$ is minimally generated by $\dim_k R_d$ monomials.

Proof. Since I is generated by generic forms, $\text{in}(I)$ equals the generic initial ideal of I , hence $\text{in}(I)$ is a strongly stable ideal by [10, Theorem 28.4]. Also, as at least $\dim_k R_d - \dim_k R_{d-1}$ of the generators of I have degree d , $\text{in}(I)$ contains all the monomials of degree d that are not divisible by x_n . Since $d \geq 2$ and $n \geq 3$, we have $\dim_k R_d - \dim_k R_{d-1} \geq n$, thus R/I is Artinian. The statement now follows from Theorem 4.1. \square

Example 4.3. Let I be an ideal in $R = k[x_1, x_2, x_3]$ generated by $\dim_k R_{11} - \dim_k R_{10} = 12$ generic forms of degree 11. Let I' be I plus the ideal generated by a generic form of degree 12, and let I'' be I plus the ideal generated by the non-generic form $(x_1 + x_2 + x_3)^5 x_3^7$. All three ideals will have socle in degree 13, and the non-trivial parts of the Betti tables equals

	1 2 3		1 2 3		1 2 3
total:	78 143 66		78 143 66		76 140 65
10 :	12 11 .		12 11 .		10 : 12 11 .
11 :	11 22 11		11 : 12 24 12		11 : 11 22 11
12 :	22 44 22		12 : 24 48 24		12 : 22 47 24
13 :	33 66 33		13 : 30 60 30		13 : 30 60 30

for $\text{in}(I)$, $\text{in}(I')$, and $\text{in}(I'')$, respectively. In particular, the total Betti numbers for $\text{in}(I)$ and $\text{in}(I')$ agree, while those for $\text{in}(I'')$ differ due to the fact that I'' is not generated solely by generic forms.

Remark 4.4. Let $I = \text{Ann}(G)$, where $G \in S$ is a generic form of degree $e \geq 1$. According to Theorem 2.1, $\text{in}(I)$ is a reverse lexicographic, hence strongly stable, ideal that is generated in degrees $\lfloor e/2 \rfloor + 1 = d + 1 \leq q \leq e + 1$ and contains all monomials of degree $d+1$ not divisible by x_n . Hence, $\text{in}(I)$ satisfies the requirements of Theorem 4.1, so that $\beta_p(R/\text{in}(I)) = \beta_p(R/\mathfrak{m}^{d+1})$ for all $p \geq 0$.

However, by Corollary 2.3, we know exactly the basis elements of $\text{in}(I)$, so we can actually specify all the graded Betti numbers of $R/\text{in}(I)$.

Since $\text{in}(I)$ is a strongly stable ideal, by [10, Corollary 28.12] we have that

$$\beta_{p,p+q}(R/\text{in}(I)) = \sum_{i=1}^n a_{i,q} \binom{i-1}{p-1}$$

for all $p \geq 1$ and $q \geq 0$, where $a_{i,q}$ denotes the number of minimal generators of $\text{in}(I)$ that have degree q and that are divisible by x_i but not by any x_j with $j > i$.

In degree $d+1$ we have that $\text{in}(I)$ contains all monomials not divisible by x_n , so that $a_{i,d+1} = \dim_k k[x_1, \dots, x_i]_d = \binom{d+i-1}{d}$ for all $i < n$ and $a_{i,q} = 0$ for all $i < n$, $q > d+1$. From Corollary 2.3 we get $a_{n,q} = \dim_k k[x_1, \dots, x_{n-1}]_{e+1-q} = \binom{e-q+n-1}{n-2}$ for all $q > d+1$. We also see that $a_{n,d+1} = 0$ if $e = 2d+1$ is odd and that $a_{n,d+1} = \dim_k k[x_1, \dots, x_{n-1}]_d = \binom{d+n-2}{d}$ if $e = 2d$ is even.

We conclude that

$$\beta_{p,p+q}(R/\text{in}(I)) = \binom{e-q+n-1}{n-2} \binom{n-1}{p-1}$$

for all $p \geq 1$ and $q > d+1$, whereas

$$\beta_{p,p+d+1}(R/\text{in}(I)) = \begin{cases} \sum_{i=1}^{n-1} \binom{d+i-1}{d} \binom{i-1}{p-1} & \text{if } e = 2d+1 \text{ is odd} \\ \binom{d+n-2}{d} + \sum_{i=1}^{n-1} \binom{d+i-1}{d} \binom{i-1}{p-1} & \text{if } e = 2d \text{ is even} \end{cases}$$

for all $p \geq 1$.

Our computational experiments also give evidence for the following.

Conjecture 4.5. Let $G \in S$ is a generic form of degree $e \geq 1$. Let I be an ideal of R generated by generic forms of degrees greater than or equal to $d = \lfloor e/2 \rfloor + 1$, of which at least $\dim_k R_d - \dim_k R_{d-1}$ have degree d . Then

$$\text{in}(\text{Ann}(G)) \subseteq \text{in}(I).$$

If I is as in Conjecture 4.5 then, by Corollary 4.2, we know that the minimal number of generators of $\text{in}(I)$ is $\dim_k R_d$. However, we do not know how many minimal generators $\text{in}(I)$ has in each degree. If that could be determined, then, as in Remark 4.4, we would know the whole Betti table of $R/\text{in}(I)$ and, as a consequence, one could possibly determine the Hilbert function of R/I and check if it matches with the Hilbert function predicted by Fröberg's conjecture.

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