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The C*-algebra of a composition reflection.

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Abstract

We study the C* algebra generated by the composition operator C_a acting on the Hardy space H^2 of the unit disk, given by $C_a f = f \circ \varphi_a$, where

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

for $|a| < 1$. Also several operators related to C_a are examined.

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1 Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the unit disk and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle. For $a \in \mathbb{D}$, consider the analytic automorphism which maps $\mathbb{D}$ onto $\mathbb{D}$

$$\varphi_a(z) = \frac{a - z}{1 - \bar{a}z},$$

The fact that $\varphi_a(\varphi_a(z)) = z$ implies that the composition operator

$$C_{\varphi_a} f = f \circ \varphi_a. \tag{1}$$

induced by this automorphisms is a *reflection* (i.e., satisfies that $C_a^2 = I$) in $H^2 = H^2(\mathbb{D})$, the Hardy space of the disk (see [6]). In this note we characterize the C*-algebra $\mathcal{C}^*(C_a)$ generated by C_a, which is also the C*-algebra generated by two projections, the ortogonal projections onto the two eigenspaces of C_a: $N(C_a - I)$ and $N(C_a + I)$. We profit from the vast bibliography on this subject, specially the results by G.K Pedersen [11], and the excellent survey [3].

We also consider several operators related to C_a. Among these, the symmetry (=self-adjoint reflections) ρ_a, obtained from the polar decomposition of C_a, and $W_a = M_{\psi_a} C_a$, where $\psi_a = \frac{(1-|a|^2)^{1/2}}{1-\bar{a}z}$ is the normalized Szego kernel.

The characterization of $\mathcal{C}^*(C_a)$ requires the study of the spectrum of the product of the projections onto $N(C_a \pm I)$.

The contents of the paper are the following. In Section 2 we introduce preliminay facts and notations. In Section 3 we study elementary relations between the spectra of of PQ and $P \pm Q$, for P, Q orthogonal projections. In Section 4 we apply these result to the case of $P = P_{N(C_a - I)}$ and $Q = P_{N(C_a + I)}$. In the short Section 5 we use these data to characterize $\mathcal{C}^*(C_a)$, using the results of G.K. Pedersen [11] (via the exposition done in [3]). The automorphism φ_a induces also a reflection in $L^2(\mathbb{T})$, which we call Γ_a; in Section 6 we study the relation between C_a and Γ_a. In Section 7 we study the mentioned symmetries ρ_a and W_a. In Section 8 we consider the relation of C_a with the Toeplitz isometry $T_{\varphi_{\omega_a}}$, where ω_a is the unique fixed point of φ_a inside $\mathbb{D}$.

2 Preliminaries and notations

Note that $C_0 f(z) = f(-z)$. Denote by $\mathcal{E}$ the subspace of *even* functions in H^2:

$$\mathcal{E} = \{f \in H^2 : f(z) = f(-z)\} = N(C_0 - I).$$

Its orthogonal complement is the space $\mathcal{O}$ of *odd* functions, $\mathcal{O} = N(C_0 + I)$. Denote by ω_a the unique fixed point of φ_a inside the disk:

$$\omega_a := \frac{1}{\bar{a}}\{1 - \sqrt{1 - |a|^2}\} \ \text{ if } a \neq 0, \quad \text{and } \omega_0 = 0.$$

The fixed point ω_a is useful in describing the two eigenspaces of C_a for $a \neq 0$. In [2] it was shown that

$$N(C_a - I) = C_{\omega_a}(\mathcal{E}) \ \text{ and } \ N(C_a + I) = C_{\omega_a}(\mathcal{O}).$$

These assertions follow in a direct manner from the elementary identity

$$\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a}. \tag{2}$$

In general ($a \neq 0$), C_a is non-selfadjoint. It was shown by Cowen [5] (see also [6]) that

$$C_a^* = (C_{\varphi_a})^* = M_{\frac{1}{1-\bar{a}z}} C_a (M_{1-\bar{a}z})^* = M_{\frac{1}{1-\bar{a}z}} C_a T_{1-a\bar{z}},$$

where, for $g \in L^\infty(\mathbb{T})$, M_g and T_g denote, respectively, the multiplication and Toeplitz operators with symbol g. Equivalently,

$$C_a^* = M_{\frac{1}{1-\bar{a}z}} C_a - a M_{\frac{1}{1-\bar{a}z}} C_a S^*, \tag{3}$$

where $S^* = (M_z)^*$ (or co-shift) is the adjoint of the shift operator $S = M_z$. Then

$$C_a C_a^* = \frac{1}{1-|a|^2}(I - \bar{a}S)(I - aS^*) = \frac{1}{1-|a|^2} T_{1-\bar{a}z} T_{1-a\bar{z}}. \tag{4}$$

On the other hand, it can be shown that

$$C_a^* C_a = (1 - |a|^2) T_{\frac{1}{|1-\bar{a}z|^2}}, \tag{5}$$

which also equals (see Proposition 7.5 in [8])

$$C_a^* C_a = (1 - |a|^2) T_{\frac{1}{1-a\bar{z}}} T_{\frac{1}{1-\bar{a}z}}. \tag{6}$$

Remark 2.1. Using (5), one easily obtains that the spectrum of $C_a^* C_a$ is

$$\sigma(C_a^* C_a) = \sigma(T_{\frac{1-|a|^2}{|1-\bar{a}z|}}) = [\frac{1-|a|}{1+|a|}, \frac{1+|a|}{1-|a|}],$$

the image of $\mathbb{T}$ under the function $\frac{1-|a|^2}{|1-\bar{a}z|}$ (see [8]). Since C_a is invertible, we also have $\sigma(C_a C_a^*)$ equals this interval. In particular,

$$\|C_a\| = \|C_a^*\| = \|C_a^* C_a\|^{1/2} = \frac{1+|a|}{1-|a|},$$

which is well know (see [6]). Similarly, we obtain the spectrum and the norm of the commutator $[C_a^*, C_a]$ (note that $C_a C_a^* = (C_a^* C_a)^{-1}$):

$$[C_a^*, C_a] = C_a^* C_a - (C_a^* C_a)^{-1} = f(C_a^* C_a),$$

where $f(t) = t - \frac{1}{t}$ (considered in $(0, +\infty)$). Then

$$\sigma([C_a^*, C_a]) = f\left([\frac{1-|a|}{1+|a|}, \frac{1+|a|}{1-|a|}] \right) = [\frac{-4|a|}{1-|a|^2}, \frac{4|a|}{1-|a|^2}],$$

and also $\|[C_a^*, C_a]\| = \frac{4|a|}{1-|a|^2}$.

3 Spectral relations between PQP and $P \pm Q$

In this section we collect several elementary (certainly well known) results concerning the spectra of PQP and $P \pm Q$ for pairs of orthogonal projections P, Q. First we state properties concerning eigenvalues. Note that PQP is a positive contraction, and that $P - Q$ is a selfadjoint contraction. Chandler Davis [7] observed that the spectrum of $P - Q$ is symmetric with respect to the oirigin, in the following sense. Denote $A = P - Q$ and put $\mathcal{H}' = (N(A - I) \oplus N(A + I))^\perp$. Then it is elementary that $\mathcal{H}' = (R(P) \cap N(Q) \oplus N(P) \cap R(Q))^\perp$ reduces both P and Q. Denote by P' and Q' (and $A' = P' - Q'$) the corresponding reductions. Then there exists a symmetry V of $\mathcal{H}$ such that $VP'V = Q'$ (and therefore also $VQ'V = P'$). Thus $VA'V = -A'$, and in particular the spectrum of A' is symmetric: $\lambda \in \sigma(A')$ iff $-\lambda \in \sigma(A')$, and the multiplicity function is symmetric. It follows that the spectrum of A has the same property, save for the eventual eigenvalues ± 1, where this symmetry could break.

Let us state the following basic properties.

Lemma 3.1. *Suppose that P and Q are orthogonal projections acting in $\mathcal{H}$.*

1. *If PQP has a an eigenvalue $\lambda \neq 0, 1$, then $\pm(1-\lambda^2)^{1/2}$ are eigenvalues of $P-Q$.*

2. *Conversely, if $\mu \neq 0, 1, -1$ is an eigenvalue of $P-Q$, then $1-\mu^2$ is an eigenvalue for PQP*

Proof. Suppose $PQPf = \lambda f$ for $\lambda \neq 0, 1$ and $\|f\| = 1$. Then $f \in R(P)$ and therefore the subspace $\mathcal{V}$ generated by f and Qf is invariant for P and Q:

$$Pf = f \; ; \; PQf = PQPf = \lambda f \quad \text{and} \quad Qf$$

belong to $\mathcal{V}$. Then $P-Q$ is a selfadjoint operator acting in $\mathcal{V}$, which is two dimensional (if Qf where a multiple of f, then either $Qf = 0$ and then $PQPf = PQf = 0$; or $Qf = \alpha f$ and thus $f \in R(P) \cap R(Q)$ and therefore $PQPf = f$). Then $\{f, Qf\}$ is a basis for $\mathcal{V}$ and the matrices of P, Q and $P-Q$ as operators in $\mathcal{V}$ for this basis are, respectively:

$$\begin{pmatrix} 1 & \lambda \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & \lambda \\ -1 & -1 \end{pmatrix}.$$

Note that $\lambda \in (0,1)$: $PQP \geq 0$. The eigenvalues of the third matrix are $-(1-\lambda^2)^{1/2}$ and $(1-\lambda^2)^{1/2}$.

For the second statement, suppose that $Pg - Qg = \mu g$, for $\mu \neq 0, \pm 1$, i.e.,

$$Qg = Pg - \mu g. \tag{7}$$

Then, applying Q one has $Qg = QPg - \mu Qg$, i.e. $Qg = \frac{1}{\mu+1} g$. Then, substituting this identity in (7), we get

$$\frac{1}{\mu+1} QPg = Pg - \mu g,$$

and applying P: $\frac{1}{\mu+1} PQPg = (1-\mu)Pg$, i.e.

$$PQPg = (1-\mu^2)Pg.$$

Note that $Pg \neq 0$: $Pg = 0$ would imply $Qg = -\mu g$, i.e. $\mu = 0$ or $\mu = -1$. □

Remark 3.2. Note that if $f \neq 0$, $PQPf = f$ if and only if $f \in R(P) \cap R(Q)$. Sufficiency is trivial. Necessity: pick $\|f\| = 1$; $PQPf = f$ implies $f \in R(P)$, and thus $PQf = f$. Then

$$1 = \langle PQf, f \rangle = \langle Qf, Pf \rangle = \langle Qf, f \rangle,$$

which clearly implies $f \in R(Q)$.

The following result can be verified along the same lines as the above lemma. We include the elementary proof.

Lemma 3.3. *Let P, Q be orthogonal projections. Then λ is an eigenvalue of $P-Q$ with $|\lambda| < 1$ if and only if $1 \pm (1-\lambda^2)^{1/2}$ are eigenvalues of $P+Q$.*

Proof. Let λ be an eigevalue of $P-Q$. First suppose that $\lambda = 0$, and $(P-Q)f = 0$ for $f \neq 0$. Then $f \in R(P) \cap R(Q) \oplus N(P) \cap N(Q)$. Thus $f = f_1 + f_0$ with $Pf_1 = f_1 = Qf_1$ and $Pf_0 = 0 = Qf_0$. Then $(P+Q)f_1 = 2f_1$ and $(P+Q)f_0 = 0$, i.e. $1 \pm (1-0)^{1/2}$ are eigenvalues of $P+Q$.

Suppose now that $(P-Q)f = \lambda f$ with $\lambda \neq -1, 0, 1$. Consider, as in the proof above, the subspace $\mathcal{V}$ generated by f and Qf. Note that $\{f, Qf\}$ are linearly independent: if $Qf = \alpha f$, then either $\alpha = 0$ or $\alpha = 1$. If $\alpha = 0$, then $\lambda f = (P-Q)f = Pf$ (and $\lambda \neq 0$) imply $\lambda = 1$ (a contradiction); if $\alpha = 1$, then $\lambda f = (P-Q)f = Pf - f$ implies $Pf = (1+\lambda)f$, i.e. $\lambda = -1$ (again a contradiction). Thus $\{f, Qf\}$ is a basis for $\mathcal{V}$. Note that $(P+Q)f = (P-Q)f + 2Qf = \lambda f + 2Qf$. On the other hand, $(P-Q)f = \lambda f$ implies that $Pf = \lambda f + Qf$, and thus $Pf = P(\lambda f + Qf) = \lambda Pf + PQf$ and thus $PQf = (1-\lambda)Pf = (1-\lambda)(\lambda f + Qf) = (\lambda - \lambda^2)f + (1-\lambda)Qf$. Then $(P+Q)Qf = PQf + Qf = (\lambda - \lambda^2)f + (2-\lambda)Qf$. Therefore $(P+Q)\mathcal{V} \subset \mathcal{V}$ and its matrix in the basis $\{f, Qf\}$ is

$$\begin{pmatrix} \lambda & \lambda - \lambda^2 \\ 2 & 2 - \lambda \end{pmatrix},$$

whose eigenvalues are $1 \pm (1-\lambda^2)^{1/2}$.

The converse is similar. $\square$

Now we focus on arbitrary spectral values, not necessarily eigenvalues. To this effect, we shall need P. Halmos [10] results on pairs of subspaces / projections. Fix P, Q orthogonal projections. consider the folowing natural orthogonal decomposition of $\mathcal{H}$:

$$R(P) \cap R(Q) \;\oplus\; N(P) \cap N(Q) \;\oplus\; R(P) \cap N(Q) \;\oplus\; N(P) \cap R(Q) \;\oplus\; \mathcal{H}_0.$$

The space $\mathcal{H}_0$ is usually called the generic part of P and Q. Clearly this decomposition reduces both P and Q. Note that in this decomposition,

$$P = I \;\oplus\; 0 \;\oplus\; I \;\oplus\; 0 \;\oplus\; P_0 \quad \text{and} \quad Q = I \;\oplus\; 0 \;\oplus\; 0 \;\oplus\; I \;\oplus\; Q_0.$$

So that

$$PQP = I \;\oplus\; 0 \;\oplus\; 0 \;\oplus\; 0 \;\oplus\; P_0 Q_0 P_0,$$
$$P - Q = 0 \;\oplus\; 0 \;\oplus\; I \;\oplus\; -I \;\oplus\; P_0 - Q_0.$$

and

$$P + Q = 2 \;\oplus\; 0 \;\oplus\; I \;\oplus\; I \;\oplus\; P_0 + Q_0.$$

Therefore, in order to analyze the spectra of PQP, $P-Q$ and $P+Q$ we need to focus on the operators acting in the generic part $\mathcal{H}_0$.

Continuing with Halmos' theory, he proved that there exists a unitary isomorphim between $\mathcal{H}_0$ and a product Hilbert space $\mathcal{L} \times \mathcal{L}$, and a positive operator X acting in $\mathcal{L}$, with $\|X\| \leq \pi/2$ and $N(X) = \{0\}$, such that the projections P_0 and Q_0 are carried to

$$P_0 \simeq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_0 \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(X)$ and $S = \sin(X)$.

Note in particular that $\sigma(X) \subset [0, \pi/2]$.

Lemma 3.4. *Let P_0 and Q_0 as above and let $\lambda \neq 0$. Then*

1.

$$\lambda \in \sigma(P_0Q_0P_0) \iff \pm\sqrt{1-\lambda^2} \in \sigma(P_0 - Q_0).$$

2.

$$\lambda \in \sigma(P_0Q_0P_0) \iff 1 \pm \lambda^{1/2} \in \sigma(P_0 + Q_0).$$

Proof. 1. We can reason with the operators in $\mathcal{L} \times \mathcal{L}$. Then

$$P_0Q_0P_0 \simeq \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix},$$

and thus $\sigma(P_0Q_0P_0) = \{\cos^2(t) : t \in \sigma(X)\} \cup \{0\}$. Also

$$P_0 - Q_0 \simeq \begin{pmatrix} S^2 & -CS \\ -CS & -S^2 \end{pmatrix} = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix},$$

where the last two matrices commute (we have used that C and S commute). Note that the second matrix is a symmetry:

$$\begin{pmatrix} S & -C \\ -C & -S \end{pmatrix}^* = \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix} \text{ and } \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix}^2 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

It is well known that if a, b are elements of a C*-algebra such that $ab = ba$, then $\sigma(ab) \subset \{\lambda\mu : \lambda \in \sigma(a), \mu \in \sigma(b)\}$. Therefore, since $\sigma\left(\begin{pmatrix} S & -C \\ -C & -S \end{pmatrix}\right) = \{-1; 1\}$, we have that

$$\sigma(P_0 - Q_0) \subset \{\pm\lambda : \lambda \in \sigma \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}\} = \{\pm \sin(t) : t \in \sigma(X)\}.$$

Conversely, suppose that $P_0 - Q_0 - \lambda 1$ is invertible. Since

$$P_0 - Q_0 - \lambda 1 \simeq \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix} \{ \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} - \lambda \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix} \},$$

this means that $\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} - \lambda \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix}$ is invertible. Thus the square of this operator

$$\begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix} - 2\lambda \begin{pmatrix} S & -C \\ -C & -S \end{pmatrix} + \lambda^2 \begin{pmatrix} S^2 & 0 \\ 0 & S^2 \end{pmatrix}$$

is positive and invertible. Therefore the diagonal entries are positive and invertible, i.e., $S^2 \pm 2\lambda S + \lambda^2 I = (S \pm \lambda I)^2$ is invertible. That is, $\lambda \neq \pm \sin(t)$ for $t \in \sigma(X)$.

2. The proof is similar. Note that

$$P_0 + Q_0 \simeq \begin{pmatrix} I + C^2 & CS \\ CS & S^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix}$$

and that

$$\begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix} = \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} C & S \\ S & -C \end{pmatrix}.$$

The right hand matrices commute, and the matrix $\begin{pmatrix} C & S \\ S & -C \end{pmatrix}$ is also a symmetry. Thus, with the same argument as above, we have that

$$\sigma\left(\begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix}\right) \subset \{\pm\cos(t) : t \in \sigma(X)\}.$$

Also note that

$$\begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix} - \lambda \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \{ \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix} - \lambda \begin{pmatrix} C & S \\ S & -C \end{pmatrix} \}.$$

This product is invertible if and only if the right hand factor is invertible, which implies that its square is positive and invertible, and therefore the diagonal entries of this square are invertible, i.e. $(C \pm \lambda I)^2$ are invertible. Therefore

$$\sigma(P_0 + Q_0) = \sigma\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} C^2 & CS \\ CS & -C^2 \end{pmatrix}\right) = \{1 \pm \cos(t) : t \in \sigma(X)\}.$$

□

4 Projections onto the eigenspaces of C_a

Recall from [1] the formulas for the orthogonal projections onto the range and the null space of an oblique projection Q:

$$P_{R(Q)} = Q(Q + Q^* - I)^{-1} \text{ and } P_{N(Q)} = (I - Q)(I - Q - Q^*)^{-1}.$$

For the oblique projection $\frac{1}{2}(I - C_a)$, whose range and nullspace are, respectively, the (non orthogonal) eigenspaces $N(C_a - I)$ and $N(C_a + I)$, we have:

$$P_{N(C_a - I)} = (I + C_a)(C_a + C_a^*)^{-1} \text{ and } P_{N(C_a + I)} = (C_a - I)(C_a + C_a^*)^{-1}. \tag{8}$$

Denote by

$$\mathbf{\Delta}_a := P_{N(C_a - I)} P_{N(C_a + I)} P_{N(C_a - I)}.$$

We shall compute the norm $\|P_{N(C_a-I)} P_{N(C_a+I)}\| = \|\mathbf{\Delta}_a\|^{1/2}$ below, and further study the spectrum of $\mathbf{\Delta}_a$.

The following result will be useful:

Theorem 4.1. *Let $a \in \mathbb{D}$. Then*

$$\sigma(P_{N(C_a-I)} + P_{N(C_a+I)}) = [\frac{2 - 2|a|^2}{2 - |a|^2}, 1 + |a|].$$

Moreover, none of these spectral values are eigenvalues.

Proof. It follows from (8) that

$$P_{N(C_a-I)} + P_{N(C_a+I)} = 2C_a(C_a + C_a^*)^{-1}.$$

Note that the inverse of the element $C_a(C_a + C_a^*)^{-1}$ is (using (5))

$$(C_a + C_a^*)C_a = I + C_a^* C_a = I + (1 - |a|^2)T_{\frac{1}{|1-\bar{a}z|^2}}.$$

The spectrum of $T_{\frac{1}{|1-\bar{a}z|^2}}$ is the image of its symbol [8]:

$$\sigma(T_{\frac{1}{|1-\bar{a}z|^2}}) = [\frac{1}{(1+|a|)^2}, \frac{1}{(1-|a|)^2}].$$

Then

$$\sigma((I + C_a^* C_a)^{-1}) = [\frac{1-|a|^2}{2-|a|^2}, \frac{1+|a|}{2}]$$

and therefore

$$\sigma(P_{N(C_a-I)} + P_{N(C_a+I)}) = [\frac{2-2|a|^2}{2-|a|^2}, 1 + |a|].$$

An eigenvalue of $P_{N(C_a-I)} + P_{N(C_a+I)}$ would yield an eigenvalue of $T_{\frac{1}{|1-\bar{a}z|^2}}$. □

In particular, we have that $\|P_{N(C_a-I)} + P_{N(C_a+I)}\| = 1 + |a|$. We can compute the norm of $\mathbf{\Delta}_a$:

Proposition 4.2. *The operator $\mathbf{\Delta}_a$ has no (non nil) eigenvalues in H^2. Moreover.*

$$\|\mathbf{\Delta}_a\| = |a|^2.$$

Proof. Let $\lambda \neq 0$ be an eigenvalue of $\mathbf{\Delta}_a$: $\mathbf{\Delta}_a f = \lambda f$, for $\|f\| = 1$. Since clearly $\|\mathbf{\Delta}_a\| \leq 1$, it must be $|\lambda| \leq 1$. Note that $\mathbf{\Delta}_a f = f$ would imply (see Remark 3.2) that $f \in N(C_a - I) \cap N(C_a + I) = \{0\}$. Thus $\lambda < 1$.

Since $f \in R(P_{N(C_a-I)})$,

$$\begin{aligned} P_{N(C_a-I)}P^\perp_{N(C_a+I)}P_{N(C_a-I)}f &= (P_{N(C_a-I)} - P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a-I)})f \\ &= (I - P_{N(C_a-I)}P_{N(C_a+I)}P_{N(C_a+I)})f = (1-\lambda)f. \end{aligned}$$

Then using Lemma 3.1 with $P = P_{N(C_a-I)}$ and $Q = P^\perp_{N(C_a+I)}$, we get that $\pm\left(1-(1-\lambda)^2\right)^{1/2}$ are eigenvalues of

$$P - Q^\perp = P_{N(C_a-I)} - P^\perp_{N(C_a+I)} = P_{N(C_a-I)} + P_{N(C_a+I)} - I.$$

But it was shown in Theorem 4.1 that $P_{N(C_a-I)} + P_{N(C_a+I)}$ has no eigenvalues.

From Theorem 4.1 we know also that $\sigma(P_{N(C_a-I)} + P_{N(C_a+I)}) = [\frac{2-2|a|^2}{2-|a|^2}, 1+|a|]$. Therefore, $\|P_{N(C_a-I)} + P_{N(C_a+I)}\| = 1+|a|$. In [9] J. Duncan and P.J. Taylor proved that if P, Q are non nil projections, then

$$\|P + Q\| = 1 + \|PQ\|.$$

Then $\|P_{N(C_a-I)}P_{N(C_a+I)}\| = |a|$. □

We can determine the spectrum of $\mathbf{\Delta}_a$ (in particular, obtain another proof of $\|\mathbf{\Delta}_a\| = |a|^2$). We shall use Theorem 3.4. Therefore, it will be useful to compute the position of $N(C_a - I)$ and $N(C_a + I)$. First note that since these subspaces are complementary,

$$N(C_a - I) \cap N(C_a + I) = \{0\}$$

and

$$N(C_a - I)^\perp \cap N(C_a + I)^\perp = \langle N(C_a - I) + N(C_a + I)\rangle^\perp = \{0\}.$$

In [2] it was shown (Prop. 6.1) that

$$\dim N(C_a - I) \cap N(C_a + I)^\perp = 1 \quad \text{but} \quad N(C_a - I)^\perp \cap N(C_a + I) = \{0\}. \tag{9}$$

Therefore in Halmos decomposition of H^2 in terms of $N(C_a - I)$ and $N(C_a + I)$ we have only two non trivial subspaces:

$$N(C_a - I) \cap N(C_a + I)^\perp \oplus \mathcal{H}_0$$

Denote $\mathbf{\Delta}_a = 0 \oplus \mathbf{\Delta}_a^0$, where $\mathbf{\Delta}_a^0$ is the reduction of $\mathbf{\Delta}_a$ to $\mathcal{H}_0$. Clearly $\sigma(\mathbf{\Delta}_a) = \sigma(\mathbf{\Delta}_a^0) \cup \{0\}$.

Proposition 4.3. $\sigma(\mathbf{\Delta}_a) = [0, |a|^2]$.

Proof. Due to the observation above, we have to compute the spectrum of the reduction $\mathbf{\Delta}_a^0$. Denote by P_0, Q_0 the reductions of $P_{N(C_a-I)}$ and $P_{N(C_a+I)}$ to $\mathcal{H}_0$. The spectrum of $P_0 + Q_0$ is obtained from

$$\sigma(P_{N(C_a-I)} + P_{N(C_a+I)}) = \{1\} \cup \sigma(P_0 + Q_0).$$

Since $\sigma(P_{N(C_a-I)} + P_{N(C_a+I)}) = [\frac{2-2|a|^2}{2-|a|^2}, 1 + |a|]$, and $\frac{2-2|a|^2}{2-|a|^2} < 1$, we have that also

$$\sigma(P_0 + Q_0) = [\frac{2-2|a|^2}{2-|a|^2}, 1 + |a|].$$

Recall from Theorem 3.4 that $\mu \in \sigma(P_0 + Q_0)$ if and only if $\lambda = (\mu - 1)^2 \in \sigma(P_0Q_0P_0)$. Then the spectrum of $\mathbf{\Delta}_a^0$ is the image of the function $f(\mu) = (\mu - 1)^2$ in the interval $[\frac{2-2|a|^2}{2-|a|^2}, 1 + |a|]$, i.e., $[0, |a|^2]$. □

It is known (and elementary to verify) that

$$N(C_a - I) \cap N(C_a + I)^\perp = N(P_{N(C_a-I)} - P_{N(C_a+I)} - I)$$

and

$$N(C_a - I)^\perp \cap N(C_a + I) = N(P_{N(C_a-I)} - P_{N(C_a+I)} + I).$$

Then using again the intersections computed in (9), we have that

Proposition 4.4. $\|P_{N(C_a-I)} - P_{N(C_a+I)}\| = 1$.

Proof. If $f \in N(C_a - I) \cap N(C_a + I)^\perp$ with $\|f\| = 1$, then $(P_{N(C_a-I)} - P_{N(C_a+I)})f = f$. □

5 The C*-algebra generated by C_a.

The C^*-algebra $\mathcal{C}^*(C_a)$ generated by C_a coincides with the C*-algebra generated by the projections $P_{N(C_a-I)}$ and $P_{N(C_a+I)}$. Therefore, by the results of G.K. Pedesersen [11] it is completely characterized by the spectrum of $\mathbf{\Delta}_a$ (see for instance the excellent survey [3] on which we will base our exposition). Following the notation in [3], given the subspaces $\mathcal{L} = N(C_a - I)$ and $\mathcal{N} = N(C_a + I)$, in order to characterize $\mathcal{C}_a^*$ we need the subspaces

$$\mathcal{M}_{00} = \mathcal{L} \cap \mathcal{N},\ \mathcal{M}_{01} = \mathcal{L} \cap \mathcal{N}^\perp,\ \mathcal{M}_{01} = \mathcal{L}^\perp \cap \mathcal{N},\ \mathcal{M}_{11} = \mathcal{L}^\perp \cap \mathcal{N}^\perp,$$

and $\mathcal{H}_0$ the orthogonal complement of the sum of the former four. We already noticed that $\mathcal{M}_{00} = \mathcal{M}_{11} = \mathcal{M}_{10} = \{0\}$ and $\dim \mathcal{M}_{01} = 1$. Again using Halmos' theory [10], we have that $\mathcal{H}_0 \simeq \mathcal{J} \times \mathcal{J}$ and there exists a positive operator $X \in \mathcal{B}(\mathcal{J})$, $X \le \pi/2$, $N(X) = \{0\}$, such that the isomorphism that carries $\mathcal{H}_0$ onto $\mathcal{J} \times \mathcal{J}$ maps the projections $P_{\mathcal{L}} = P_{N(C_a-I)}$ and $P_{\mathcal{N}} = P_{N(C_a+I)}$ onto

$$P_{N(C_a-I)} \simeq \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \ \text{ and }\ P_{N(C_a+I)} \simeq \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(X)$ and $S = \sin(X)$. Then $\mathcal{C}^*(C_a)$ can be described in terms of $H = \sin(X)^2$, or more precisely, in terms of the spectrum of H. Since $\sigma(\mathbf{\Delta}_a) = [0, |a|]$, i.e., $\sigma(X) = [\arccos(|a|), \pi/2]$. It follows that $\sigma(H) = [1 - |a|^2, 1]$. Then according to Theorem 4.1 in [3], we have that

Theorem 5.1. *Let $a \in \mathbb{D}$, $a \ne 0$. Then $\mathcal{C}_a^*$ is $*$-isomorphic to*

$$\{(\alpha, \begin{pmatrix} f_{00}(H) & f_{01}(H) \\ f_{10}(H) & f_{11}(H) \end{pmatrix}) : \alpha \in \mathbb{C}, f_{ij} \in C(1 - |a|^2, 1), f_{00}(1) = \alpha, f_{01}(1) = f_{10}(1) = 0\}.$$

With this description, it is clear that

Corollary 5.2. *Let $a, b \in \mathbb{D} \setminus \{0\}$. Then $\mathcal{C}^*(C_a) \simeq \mathcal{C}^*(C_b)$.*

6 Relationship between C_a and Γ_a

Since φ_a is also a homeomorphism in $\mathbb{T}$, it induces a composition operator Γ_a in $L^2(\mathbb{T})$. This operator is also reflection, and is easier to handle. For instance, its adjoint is easier to compute.

Clearly $\Gamma_a|_{H^2} = C_a$. Moreover, $P_{N(\Gamma_a-I)}$ leaves H^2 invariant. Indeed, if $f \in H^2$ and $f = f_+ + f_-$ with $f_+ \in N(\Gamma_a - I)$ and $f_- \in N(\Gamma_a + I)$, then $\Gamma_a f = f_+ - f_- \in H^2$. Then $f + \Gamma_a f = 2f_+ \in H^2$, i.e., $f_+, f_- \in H^2$. Then

$$P_{N(\Gamma_a-I)}|_{H^2} = P_{N(C_a-I)} \ \text{ and }\ P_{N(\Gamma_a+I)}|_{H^2} = P_{N(C_a+I)}. \qquad (10)$$

Denote by P_+ the orthogonal projection of $L^2(\mathbb{T})$ onto H^2, and by $H_- := L^2(\mathbb{T}) \ominus H^2$. The fact that $\Gamma_a|_{H^2} = C_a$, means $P_+\Gamma_a P_+ = \Gamma_a P_+ = C_a$. On the other hand, by a change of variables argument it is easy to see that

$$\Gamma_a^* = (1-|a|^2) M_{\frac{1}{|1-\bar{a}z|^2}} \Gamma_a = (1-|a|^2) M_{\frac{1}{1-\bar{a}z}\frac{z}{z-a}} \Gamma_a.$$

If $f, g \in H^2$,

$$\langle \Gamma_a^* f, g\rangle = \langle f, \Gamma_a g\rangle = \langle f, C_a g\rangle = \langle C_a^* f, g\rangle,$$

i.e., $P_+\Gamma_a^* P_+ = C_a^*$. Let us see how Γ_a^* acts on H^2 (i.e., let us compute $P_+^\perp \Gamma_a^* P_+$).

Lemma 6.1. *Let $h \in H^2$, $a \in \mathbb{D}$, $a \neq 0$, and denote by $h_0 = h - h(0)$. Then $\frac{h_0(\varphi_a(z))}{z-a} \in H^2$ and*

$$\Gamma_a^* h = (1-|a|^2)\frac{z}{1-\bar{a}z}\frac{h_0(\varphi_a(z))}{z-a} + h(0)\Gamma_a^*(1).$$

Moreover,

$$\Gamma_a^*(1) = \frac{1}{1-\bar{a}z} + \frac{a}{z-a},$$

where the first summand lies in H^2 and the second in H_-.

Proof. If $h = h_0 + h(0)$, then

$$\Gamma_a^* h = \Gamma_a^* h_0 + h(0)\Gamma_a^*(1) = (1-|a|^2) M_{\frac{1}{1-\bar{a}z}\frac{z}{z-a}} h_0(\varphi_a(z)) + h(0)\Gamma_a^*(1).$$

Note that $h_0(\varphi_a(a)) = h_0(0) = 0$, and thus $\frac{h_0(\varphi_a(z))}{z-a} \in H^2$. On the other hand, if $n \geq 0$,

$$\langle z^n, \Gamma_a^*(1)\rangle = \langle \Gamma_a(z^n), 1\rangle = \langle \left(\frac{a-z}{1-\bar{a}z}\right)^n, 1\rangle = \left(\frac{a-z}{1-\bar{a}z}\right)^n |_{z=0} = a^n,$$

i.e. $\langle \Gamma_a^*(1), z^n\rangle = \bar{a}^n$, for $n \geq 0$, and thus (for $z \in \mathbb{T}$)

$$P_+(\Gamma_a^*(1)) = \sum_{n=0}^{\infty} \bar{a}^n z^n = \frac{1}{1-\bar{a}z}.$$

For $m < 0$,

$$\langle \Gamma_a^*(1). z^m\rangle = \langle 1, \left(\frac{a-z}{1-\bar{a}z}\right)^m\rangle = \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{\bar{a}-\bar{z}}{1-a\bar{z}}\right)^m dz = \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{\bar{a}-1/z}{1-a/z}\right)^m dz$$

$$= \frac{1}{2\pi}\int_{\mathbb{T}} \left(\frac{\bar{a}-z}{1-\bar{a}z}\right)^{-m} dz = \langle \varphi_a(z)^{-m}, 1\rangle = a^{-m}.$$

Then

$$P_+^\perp(\Gamma_a^*(1)) = \sum_{m<0} a^{-m} z^m = \sum_{m<0} \left(\frac{a}{z}\right)^{-m} = \frac{a/z}{a-a/z} = \frac{a}{z-a}.$$

□

In other words, $P_+^\perp \Gamma_a^* P_+$ is the rank one operator

$$P_+^\perp \Gamma_a^* P_+ f = \frac{a}{z-a}\langle f, 1\rangle. \tag{11}$$

Therefore, to complete the 2×2 matrix of Γ_a in terms of the decomposition $L^2(\mathbb{T}) = H^2 \oplus H_-$ it remains to compute $P_+^\perp \Gamma_a P_+^\perp$. Denote by V the symmetry of $L^2(\mathbb{T})$ given by

$$Vf(z) = f(\bar{z}).$$

Clearly $V^* = V^{-1} = V$, and V maps H_- onto $H^2 \ominus \langle 1\rangle$. Also it is clear that if $g \in H_-$, then

$$\Gamma_a(g) = VC_{\bar{a}}Vg.$$

Indeed, if $g = \sum_{k=1}^\infty a_k z^{-k}$, then

$$VC_{\bar{a}}Vg = VC_{\bar{a}}(\sum_{k=1}^\infty a_k z^k) = V(\sum_{k=1}^\infty a_k \left(\frac{\bar{a}-z}{1-az}\right)^k) = \sum_{k=1}^\infty a_k \left(\frac{\bar{a}-1/z}{1-a/z}\right)^k$$

$$= \sum_{k=1}^\infty a_k \left(\frac{a-z}{1-\bar{a}z}\right)^{-k} = \Gamma_a g.$$

In particular, note that $\Gamma_a h_- \subset H_- \oplus \langle 1\rangle$. For $g \in H_-$, the component of $\Gamma_a h$ in $\langle 1\rangle$ is (accordingly)

$$\langle \Gamma_a g, 1\rangle = \langle VC_{\bar{a}}Vh, 1\rangle = \langle C_{\bar{a}}Vg.V1\rangle = \langle C_{\bar{a}}Vg, 1\rangle,$$

recall that for $f \in H^2$, $\langle C_b f, 1\rangle = f(b)$, the quantity above equals

$$Vg(\bar{a}) = \langle Vg, k_{\bar{a}}\rangle = \langle g, Vk_{\bar{a}}\rangle = \langle g, \frac{z}{z-a}\rangle.$$

Then, we have

Theorem 6.2. *The 2×2 matrix of Γ_a in terms of the decomposition $L^2(\mathbb{T}) = H^2 \oplus H_-$ is*

$$\begin{pmatrix} C_a & \langle\ \ , \frac{a}{z-a}\rangle 1 \\ & \\ 0 & VC_{\bar{a}}V - \langle\ \ , \frac{a}{z-a}\rangle 1 \end{pmatrix}.$$

Proof. The adjoint of $\langle\ \ , 1\rangle \frac{a}{a-z}$ is $\langle\ \ , \frac{a}{a-z}\rangle 1$. □

7 Two symmetries

We shall consider two symmetries (i.e., selfadjoint unitaries) which are closely related to C_a. The first one comes from the polar decomposition of C_a:

$$C_a = \rho_a |C_a| = \rho_a (C_a^* C_a)^{1/2}, \text{ i.e., } \rho_a = C_a (C_a^* C_a)^{-1/2}.$$

In [4] it was shown that the unitary part in the polar decomposition of a reflection is in fact a symmetry. Thus, $\rho_a^* = \rho_a^{-1}$. Note then that $C_a^* = |C_a|\rho_a$ and thus

$$\rho_a C_a \rho_a = C_a^*. \tag{12}$$

The second symmetry can be found in the book [6] of C. Cowen and B. McCluer (Exercise 2.1.9): $W_a : H^2 \to H^2$,

$$W_a f = \sqrt{1-|a|^2} M_{k_a} C_a f, \tag{13}$$

where $k_a(z) = \frac{1}{1-\bar{a}z}$ is the Szego kernel, i.e.,

$$W_a f(z) = \frac{\sqrt{1-|a|^2}}{1-\bar{a}z} f(\varphi_a(z)).$$

Cowen and McCluer mention that W_a is isometric for all p norms, it is easy to see that $W_a^2 = I$. Denote by f_a the H^∞ function

$$f_a := \frac{1+\bar{\omega}_a z}{1-\bar{\omega}_a z}. \tag{14}$$

Note that $\bar{f}_a = \frac{z+\omega_a}{z-\omega_a}$ Then

Proposition 7.1. *Let $a \in \mathbb{D}$, and ω_a the fixed point of φ_a inside $\mathbb{D}$. Then*

$$W_{\omega_a} C_a W_{\omega_a} = T_{f_a} C_0 = C_0 T_{1/f_a}.$$

Proof. By direct computation:

$$\begin{aligned} W_{\omega_a} C_a W_{\omega_a} f(z) &= \sqrt{1-|\omega_a|^2} W_{\omega_a} C_a \left(f(\varphi_{\omega_a}(z)) k_{\omega_a}(z) \right) \\ &= \sqrt{1-|\omega_a|^2} W_{\omega_a} \left(f(\varphi_{\omega_a}(\varphi_a(z))) k_{\omega_a}(\varphi_a(z)) \right). \end{aligned}$$

Recall from (2) that $\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a}$. Thus, the above expression equals

$$(1-|\omega_a|^2) f(-\varphi_{\omega_a}(\varphi_{\omega_a}(z))) k_{\omega_a}(z) k_{\omega_a}(\varphi_a(\varphi_{\omega_a}(z))) = (1-|\omega_a|^2) f(-z) k_{\omega_a}(z) k_{\omega_a}(\varphi_a(\varphi_{\omega_a}(z)))$$

Note that $\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a}$ and $\varphi_{\omega_a} \circ \varphi_{\omega_a}(z) = z$ imply that $\varphi_{\omega_a} \circ \varphi_a \circ \varphi_{\omega_a}(z) = -z$, and thus

$$\varphi_a \circ \varphi_{\omega_a}(z) = \varphi_{\omega_a}(-z).$$

Then

$$k_{\omega_a}(\varphi_a(\varphi_{\omega_a}(z))) = \frac{1}{1-\bar{\omega}_a \left(\frac{\omega_a + z}{1+\bar{\omega}_a z} \right)} = \frac{1+\bar{\omega}_a z}{1-|\omega_a|^2}.$$

Therefore

$$W_{\omega_a} C_a W_{\omega_a} f(z) = f(-z) k_{\omega_a}(z)(1+\bar{\omega}_a z) = f_a(z) f(-z) = T_{f_a} C_0 f(z).$$

The facts that $(W_{\omega_a} C_a W_{\omega_a})^2 = I$, and that f_a is invertible in H^∞ imply that

$$T_{f_a} C_0 = (T_{f_a} C_0)^{-1} = C_0 T_{f_a}^{-1} = C_0 T_{1/f_a}.$$

□

Remark 7.2. Since $f_a, 1/f_a \in H^\infty$, we could write $M_{f_a}, M_{1/f_a}$ instead of $T_{f_a}, T_{1/f_a}$. Also note that

$$W_{\omega_a} C_a^* W_{\omega_a} = (W_{\omega_a} C_a W_{\omega_a})^* = (T_{f_a} C_0)^* = C_0 T_{\bar{f}_a}.$$

And similarly,

$$W_{\omega_a} C_a^* W_{\omega_a} = (C_0 T_{1/f_a})^* = T_{1/\bar{f}_a} C_0.$$

In Lemma 4.1 of [2] it was shown that the (unitary) product $W_a \rho_a$ commutes with $C_a^* C_a$ (and thus also with its inverse $C_a C_a^*$). As a consequence we obtain the following:

Proposition 7.3. *With the current notations we have that*

$$W_a \rho_a C_a = C_a (W_a \rho_a)^*, \; W_a \rho_a C_a^* = C_a^* (W_a \rho_a)^*,$$

and

$$C_a^* C_a W_a = W_a (C_a^* C_a)^{-1}.$$

Proof. Since $W_a \rho_a$ commutes with $C_a^* C_a$, it also commutes with its square root $|C_a|$ (and therefore also $\rho_a W_a = (W_a \rho_a)^*$ commutes with $|C_a|$). Then

$$W_a \rho_a C_a = W_a \rho_a \rho_a |C_a| = W_a |C_a| = \rho_a (\rho_a W_a) |C_a| = \rho_a |C_a| \rho_a W_a = C_a \rho_a W_a.$$

Thus, aking adjoints,

$$C_a^* \rho_a W_a = (W_a \rho_a C_a)^* = (C_a \rho_a W_a)^* = W_a \rho_a C_a^*.$$

Finally, using that ρ_a inertwines C_a with C_a^*,

$$C_a^* C_a W_a \rho_a = W_a \rho_a C_a^* C_a = W_a C_a \rho_a C_a = W_a C_a C_a^* \rho_a,$$

and thus $C_a^* C_a W_a = W_a C_a C_a^* = W_a (C_a^* C_a)^{-1}$. □

8 The relationship with $T_{\varphi_{\omega_a}}$

Since φ_{ω_a} is an inner function, the Toeplitz operator $T_{\varphi_{\omega_a}}$ is an isometry. It is easy to see that it has co-rank 1. Below we compute the orthogonal projection $T_{\varphi_{\omega_a}} T^*_{\varphi_{\omega_a}} = T_{\varphi_{\omega_a}} T_{\overline{\varphi}_{\omega_a}}$. We shall use the following computation:

Lemma 8.1. *Let $f \in H^2$ and $b \in \mathbb{D}$. Then*

$$P_+(f/\varphi_b) = \frac{f - f(b)}{\varphi_b} + f(b)\bar{b}.$$

Proof. Note that $f/\varphi_b = \frac{f - f(b)}{\varphi_b} + \frac{f(b)}{\varphi_b}$, where the first summand belongs to H^2. Also note that $\frac{1}{b-z} \in H_-$, and thus

$$P_+(f(b)/\varphi_b) = f(b) P_+(\frac{1 - \bar{b}z}{b - z}) = f(b) P_+(\frac{1 - |b|^2}{b - z} + \bar{b}) = f(b)\bar{b}.$$

□

Therefore, $I - T_{\varphi_{\omega_a}} T_{\overline{\varphi}_{\omega_a}}$ is the orthogonal projection onto the line generated by k_{ω_a}: using Lemma 8.1

$$T_{\varphi_{\omega_a}} T_{\overline{\varphi}_{\omega_a}} f = T_{\varphi_{\omega_a}} P_+(f/\varphi_{\omega_a}) = \varphi_{\omega_a} \left(\frac{f - f(\omega_a)}{\varphi_{\omega_a}} + f(\omega_a)\bar{\omega}_a \right) = f - f(\omega_a) + \bar{\omega}_a f(\omega_a) \varphi_{\omega_a}$$
$$= f + \langle f, k_{\omega_a} \rangle (1 - \bar{\omega}_a \varphi_{\omega_a}).$$

Note that $1 - \omega_a \varphi_{\omega_a} = \frac{1-|\omega_a|^2}{1-\bar{\omega}_a z}$, and thus the computation above equals

$$f - \langle f, k_{\omega_a} \rangle (1 - |\omega_a|^2) k_{\omega_a} = f - \langle f, (1-|\omega_a|^2)^{1/2} k_{\omega_a} \rangle (1 - |\omega_a|^2)^{1/2} k_{\omega_a} = f - \langle f, \psi_a \rangle \psi_a,$$

where $\psi_a = (1 - |\omega_a|^2)^{1/2} k_{\omega_a}$ is the normalization of k_{ω_a}.

Note the folllowing facts:

Proposition 8.2. *Let $a \in \mathbb{D}$. The Toeplitz operator $T_{\varphi_{\omega_a}}$ satisfies that*

$$T_{\varphi_{\omega_a}} C_a + C_a T_{\varphi_{\omega_a}} = 0$$

and

$$T_{\varphi_{\omega_a}} C_a^* + C_a^* T_{\varphi_{\omega_a}} = 2\omega_a \langle \quad , 1 \rangle k_{\omega_a}.$$

Proof. The first assertion is a direct computation:

$$T_{\varphi_{\omega_a}} C_a f = P_+ \left(\varphi_{\omega_a} f(\varphi_a) \right) = \varphi_{\omega_a} f(\varphi_a),$$

whereas

$$C_a T_{\varphi_{\omega_a}} f = C_a(\varphi_{\omega_a} f) = \varphi_{\omega_a}(\varphi_a) f(\varphi_a),$$

and the assertion follows recalling from (2) that $\varphi_{\omega_a} \circ \varphi_a = -\varphi_{\omega_a}$.

With respect to the second assertion, using Lemma 8.1

$$T_{1/\varphi_{\omega_a}} C_a f = P_+ \left(1/\varphi_{\omega_a} f(\varphi_a) \right) = \frac{f(\varphi_a) - f(\omega_a)}{\varphi_{\omega_a}} + \bar{\omega}_a f(\omega_a).$$

On the other hand, similarly as above

$$C_a T^*_{\varphi_{\omega_a}} f = C_a P_+ \left(1/\varphi_{\omega_a} f \right) = C_a \left(\frac{f(z) - f(\omega_a)}{\varphi_{\omega_a}} + \bar{\omega}_a f(\omega_a) \right).$$

Since $C_a(1) = 1$ and again using (2), we get

$$T_{1/\varphi_{\omega_a}} C_a f = -\frac{f(\varphi_a) - f(\omega_a)}{\varphi_{\omega_a}} + \bar{\omega}_a f(\omega_a).$$

Therefore

$$(T^*_{\varphi_{\omega_a}} C_a + C_a T^*_{\varphi_{\omega_a}}) f = 2\bar{\omega}_a f(\omega_a) = 2\bar{\omega}_a \langle f, k_{\omega_a} \rangle 1,$$

i.e.,

$$T^*_{\varphi_{\omega_a}} C_a + C_a T^*_{\varphi_{\omega_a}} = 2\bar{\omega}_a \langle \quad , k_{\omega_a} \rangle 1,$$

and thus

$$T_{\varphi_{\omega_a}} C_a^* + C_a^* T_{\varphi_{\omega_a}} = 2\omega_a \langle \quad , 1 \rangle k_{\omega_a}.$$

□

Remark 8.3. It follows form the previous lemma that $T^2_{\varphi_{\omega_a}} = T_{\varphi^2_{\omega_a}}$ commutes with C_a. Whereas

$$T^2_{\varphi_{\omega_a}} C_a^* = T_{\varphi_{\omega_a}}\left(-C_a^* T_{\varphi_{\omega_a}} + 2\omega_a\langle\ ,1\rangle k_{\omega_a}\right) = C_a^* T^2_{\varphi_{\omega_a}} + 3\omega_a\{\langle\ ,1\rangle T_{\varphi_{\omega_a}} k_{\omega_a} - \langle\ ,T^*_{\varphi_{\omega_a}}1\rangle k_{\omega_a}\}.$$

Note that $T_{\varphi_{\omega_a}} k_{\omega_a} = \frac{\omega_a - z}{(1-\bar\omega_a z)^2}$ and $T^*_{\varphi_{\omega_a}} 1 = \bar\omega_a$ (constant function). In particular $T^2_{\varphi_{\omega_a}}$ commutes with C_a^* modulo a rank two operator.

Remark 8.4. Another consequence of the above lemma is that the isomery $T_{\varphi_{\omega_a}}$ intertwines C_a with $-C_a$:

$$T^*_{\varphi_{\omega_a}} C_a T_{\varphi_{\omega_a}} f = T^*_{\varphi_{\omega_a}}\left(-\varphi_{\omega_a} f(\varphi_a)\right) = -P_+\left((1/\varphi_{\omega_a})\varphi_{\omega_a} f(\varphi_a)\right) = -f(\varphi_a),$$

i.e., $T^*_{\varphi_{\omega_a}} C_a T_{\varphi_{\omega_a}} = -C_a$.

Denote A the operator $A := W_{\omega_a} C_a W_{\omega_a} = T_{f_a} C_0 = C_0 T_{1/f_a}$ from the previous section. In [2] Lemma 4.1 (or perform the elementary computation) it was shown that for $b \in \mathbb{D}$

$$W_b T_{\varphi_b} W_b = S = T_z. \tag{15}$$

We can combine this with Proposition 8.2 to obtain

Corollary 8.5. *With the current notations, we have that*

$$SA + AS = 0$$

and

$$SA^* + A^*S = 2\omega_a\langle\ ,k_{\omega_a}\rangle 1.$$

Proof.

$$\begin{aligned}0 = W_{\omega_a}\left(T_{\varphi_{\omega_a}} C_a + C_a T_{\varphi_{\omega_a}}\right) W_{\omega_a} &= W_{\omega_a} T_{\varphi_{\omega_a}} W_{\omega_a} W_{\omega_a} C_a W_{\omega_a} + W_{\omega_a} C_a W_{\omega_a} W_{\omega_a} T_{\varphi_{\omega_a}} W_{\omega_a}\\ &= SA + AS.\end{aligned}$$

Similarly

$$SA^* + A^*S = W_{\omega_a}\left(2\omega_a\langle\ ,1\rangle k_{\omega_a}\right) W_{\omega_a} = 2\omega_a\langle\ ,W_{\omega_a}1\rangle W_{\omega_a} k_{\omega_a}.$$

The proof follows noting that $W_{\omega_a}1 = \sqrt{1-|\omega_a|^2}\, T_{k_{\omega_a}} C_{\omega_a} 1 = \sqrt{1-|\omega_a|^2}\, k_{\omega_a}$ and thus

$$1 = W_{\omega_a} W_{\omega_a} 1 = W_{\omega_a}(\sqrt{1-|\omega_a|^2}\, k_{\omega_a}) = \sqrt{1-|\omega_a|^2}\, W_{\omega_a} k_{\omega_a},$$

i.e., $W_{\omega_a} k_{\omega_a} = \frac{1}{\sqrt{1-|\omega_a|^2}} 1$. □

Note that the second assertion of the above corollary is equivalent to

$$S^*A + AS^* = 2\bar\omega_a(\langle\ ,k_{\omega_a}\rangle 1)^* = 2\bar\omega_a\langle\ ,1\rangle k_{\omega_a}. \tag{16}$$

Then, using the again this first assertion,

$$SS^*A + SAS^* = SS^*A - ASS^* = 2\bar\omega_a\langle\ ,1\rangle S k_{\omega_a}.$$

Since $SS^* = I - \langle\ ,1\rangle 1$, we get that

$$A\langle\ ,1\rangle 1 - \langle\ ,1\rangle 1A = \langle\ ,1\rangle A1 - \langle\ ,A1\rangle 1 = 2\bar\omega_a\langle\ ,1\rangle S k_{\omega_a}.$$

Since $A1 = T_{f_a} C_0 1 = f_a$, we obtain

$$\langle\ ,f_a\rangle 1 - \langle\ ,1\rangle f_a = 2\bar\omega_a\langle\ ,1\rangle S k_{\omega_a}. \tag{17}$$

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